

ON THE STRUCTURE OF THE SINGULAR SET FOR THE KINETIC FOKKER-PLANCK EQUATIONS IN DOMAINS WITH BOUNDARIES.

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ABSTRACT. We describe the structure of solutions of the kinetic Fokker-Planck equations in domains with boundaries near the singular set in one-space dimension. We study in particular the behaviour of the solutions of this equation for inelastic boundary conditions which are characterized by means of a coefficient r describing the amount of energy lost in the collisions of the particles with the boundaries of the domain. A peculiar feature of this problem is the onset of a critical exponent r_c which follows from the analysis of McKean (cf. [45]) of the properties of the stochastic process associated to the Fokker-Planck equation under consideration. In this paper, we prove rigorously that the solutions of the considered problem are nonunique if $r < r_c$ and unique if $r_c < r \leq 1$. In particular, this nonuniqueness explains the different behaviours found in the physics literature for numerical simulations of the stochastic differential equation associated to the Fokker-Planck equation. We review also the available results for the solutions of this equation with absorbing boundary conditions.

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1. INTRODUCTION

A general feature of several kinetic equations is the fact that their solutions in a domain Ω with boundaries cannot be infinitely differentiable at the points of the phase space (x, v) for which $x \in \partial\Omega$ and v is tangent to $\partial\Omega$, even if the initial data are arbitrarily smooth. This set of points is usually denoted as singular set. This property of the solutions of kinetic equations was found by Guo in his study of the Vlasov-Poisson system (cf. [21]).

In the case of the one-species Vlasov-Poisson system in 3-dimensional bounded domains with specular reflection boundary conditions, the convexity of the boundary plays a crucial role in determining the regularity of solutions (cf. [27], [28]). If the boundary of the domain is flat or convex, then one can show that a trajectory on the singular set is separated from regular one in the interior, so that classical solutions exist globally in time and are C^1 in phase space variables except on the singular set.

However, if a domain is nonconvex, then even in the Vlasov equation without interactions between the particles and with inflow boundary condition, a classical C^1 solution fails to exist in general (cf. [20]).

The lack of smoothness of solutions of the Vlasov-Poisson system is closely related to the fact that the flow describing the evolution of the characteristic curves is not a C^2 function at the singular set, something which has been observed in the dynamics of billiards, even if in this case there are not gravitational or electrical fields (cf. [12], [51]).

In the case of the Boltzmann equation near Maxwellian, under inflow, diffuse, specular reflection and bounce-back boundary conditions, it is shown in [23] that if a domain is strictly convex, then the solution is (weighted) C^1 away from the singular set. On the other hand, if there is a concave boundary point, one can construct an initial condition which induces a discontinuous solution in any given time interval (cf. [35]).

A type of equation in which the regularizing effects can be expected to be stronger than those mentioned above among the kinetic equations are the Fokker-Planck equations. These equations contain second order operators in some of the variables (the velocities), but due to the specific form of the transport terms in the remaining variables (positions) the solutions of these equations are smooth in all their variables. This issue has been extensively studied in the mathematics literature (cf. for instance [25], [49] as well as additional references in Section 12 of Chapter 8). On the other hand boundary value problems for the Fokker Planck equation have been considered in many physical situations, including diffusion controlled reactions and the dynamics of semiflexible polymers (cf. [3], [4]) and in general in problems involving the dynamics of Brownian motion described by Uhlenbeck-Ornstein processes in the presence of boundaries (cf. for instance [38], [42], [43]). From a PDE point of view, the solutions are expected to be more regular due to the smoothing effect of the random force. This was clarified by Bouchut in the case of the 3-dimensional whole space problem for the Vlasov-Poisson-Fokker-Planck equations [11].

However, in the presence of the boundary, there had been not much improvement for a few decades.

It is interesting to remark that the onset of singular behaviours at the singular set takes place even for kinetic equations associated to hypoelliptic operators. As indicated above, one of the best known hypoelliptic operators is the one associated to the kinetic Fokker-Planck equation, also known as Kolmogorov equation:

$$(1.1) \quad \partial_t P + v \cdot \nabla_x P = \Delta_v P \quad , \quad P = P(x, v, t)$$

where $t \in (0, T) \subset \mathbb{R}$ and $(x, v) \in U \subset \mathbb{R}^N \times \mathbb{R}^N$ for some $N \geq 1$ and some domain U . It is well known that any solution of (1.1) is $C^\infty((0, T) \times U)$ in spite of the fact that the second order operator ∂_{vv} acts only in the variable v (cf. [25]). However, it turns out that in domains Ω with boundaries, singular sets in the sense of kinetic equations arise for

some classes of boundary conditions. More precisely, there are some classes of boundary conditions, which arise naturally from the physical interpretation of the equation (1.1) for which the hypoellipticity property can be extended to a large fraction of the boundary ∂U , but it fails along some subsets of ∂U . In those sets the solutions are just Hölder continuous. One example of this situation is the following. Suppose that we restrict the analysis to the case $N = 1$ and that we solve (1.1) in the domain $U = (0, 1) \times \mathbb{R}$. In these cases the singular set reduces either to the two points $\{(0, 0), (1, 0)\}$. Suppose that we study this problem with the absorbing boundary condition:

$$(1.2) \quad P(0, v, t) = 0, \text{ for } v > 0, \quad P(1, v, t) = 0, \text{ for } v < 0, \quad t > 0.$$

The problem (1.1), (1.2) has been studied in [31] where it has been proved that the solutions are at most Hölder continuous in a neighbourhood of the singular points $\{(0, 0), (1, 0)\}$. A more detailed discussion about the problem with absorbing boundary conditions will be given in Chapter 30.

The main results of this paper are obtained for a different type of boundary conditions. More precisely we will consider the following problem:

$$(1.3) \quad \partial_t P + v \partial_x P = \partial_{vv} P, \quad P = P(x, v, t), \quad x > 0, \quad v \in \mathbb{R}, \quad t > 0$$

with the following boundary condition:

$$(1.4) \quad P(0, -v, t) = r^2 P(0, rv, t), \quad v > 0, \quad t > 0$$

where $0 < r < \infty$. We are particularly interested in the case in which $r \leq 1$. Notice that the physical meaning of (1.4) is that the particles arriving to the wall $\{x = 0\}$ with a velocity $-v$ bounce back to the domain $\{x > 0\}$, with a new velocity v . Notice that in the case $r = 1$ the particles bounce elastically, but for $r < 1$, the collisions are inelastic and the particles lose a fraction of their energy in the collisions. We will solve (1.3), (1.4) with the initial condition:

$$(1.5) \quad P(x, v, 0) = P_0(x, v)$$

where P_0 is a probability measure in $(x, v) \in \mathbb{R}^+ \times \mathbb{R}$.

In this case the singular point is $(x, v) = (0, 0)$. We are interested in the construction of measure valued solutions of the problem (1.3)-(1.5). It turns out that for some choices of r we might have $\int_{\{(0,0)\}} P(dx dv, t) > 0$ for $t > 0$. Moreover, $P(x, v, t)$ is a C^∞ function for $(x, v) \neq (0, 0)$, but it would be only possible to obtain uniform estimates in some suitable Hölder norms if (x, v) is close to the singular point. Actually we will obtain information about the behaviour of the solutions near the singular point for the adjoint problem of (1.3), (1.4) which proves that the solutions of this problem are at most Hölder continuous.

It is possible to give the following physical motivation to the problem (1.3)-(1.5). Suppose that we have a particle $X(t)$ which moves in the half-line $\{x > 0\}$. We will assume that, as long as $X(t) > 0$, the dynamics of the particle is similar to that of a Brownian particle, but that, every time that the particle reaches $x = 0$, it bounces inelastically, that is, the absolute value of its velocity is multiplied by a restitution coefficient r , with $0 < r \leq 1$. Formally we can state that the dynamics of the particle is given by the following set of equations:

$$(1.6) \quad \frac{dX(t)}{dt} = V(t), \quad \frac{dV(t)}{dt} = \sqrt{2}\eta(t), \quad \text{if } X(t) > 0,$$

$$(1.7) \quad X(t_*^-) = 0, \quad V(t_*^-) < 0 \implies X(t_*^+) = 0, \quad V(t_*^+) = -rV(t_*^-)$$

where $\eta(t)$ is the white noise stochastic process, which formally satisfies $\langle \eta(t) \rangle = 0$, $\langle \eta(t_1) \eta(t_2) \rangle = \delta(t_1 - t_2)$, where $\langle \cdot \rangle$ denotes average. It is not a priori obvious if the process defined by means of the equations (1.6), (1.7) can be given a precise meaning. However, standard arguments of the Theory of Stochastic Processes suggest that the probability of finding the

particle $(X(t), V(t))$ in a region of the phase space (x, v) is given by the probability density $P(x, v, t)$.

The first mathematical results for the problem (1.6), (1.7) were obtained by McKean (cf. [45]). More precisely, [45] contains in particular the probability distribution of hitting times and hitting points, defined by means of the solutions of (1.6) with initial conditions $X(0) = 0$, $V(0) = b > 0$. The hitting times are defined as $t_1 = \max\{t > 0 : X(s) > 0 \text{ for } 0 < s < t\}$ and the hitting points are defined by means of $h_1 = |V(t_1)|$. Notice that a rescaling argument allows us to reduce the computation of the joint distribution of (t_1, h_1) for $b > 0$ to the case $b = 1$. Suppose that we denote as $\{t_n\}$ the sequence of consecutive times where the solutions of (1.6), (1.7) with initial values $X(0) = X_0 \geq 0$, $V(0) = V_0$ satisfy $X(t_n) = 0$. Then, the results in [45] imply that $X(t_n) \rightarrow 0$ as $n \rightarrow \infty$, $t_n \rightarrow T < \infty$ with probability one if $r < r_c$, where

$$(1.8) \quad r_c = \exp\left(-\frac{\pi}{\sqrt{3}}\right)$$

On the contrary, the results in [45] imply that for $r \geq r_c$ we have $(X(t), V(t)) \neq 0$ for any $t \geq 0$. Details about these results can be found in [33], [34].

The results indicated above suggest that the solutions of (1.3)-(1.5) should have a very different behaviour for $r < r_c$ and $r \geq r_c$. Indeed, in the first case the equations (1.3)-(1.5) cannot be expected to define the evolution of P without some additional information about the behaviour of the solutions at the singular point $(x, v) = (0, 0)$. On the contrary, the problem (1.3)-(1.5) should be able to define a unique dynamics for P if $r \geq r_c$. The main goal of this paper is to show that we can have different evolutions for the problem (1.3)-(1.5) if $r < r_c$. On the contrary, we will prove that for $r > r_c$ no additional information is required in order to define the unique evolution of the problem.

The problem (1.6), (1.7) has been also considered in the physics literature [14, 17, 7, 8, 38, 3]. The paper [14] arrives to conclusions analogous to those which follow from McKean's results by means of the analysis of some particular solutions of the Fokker-Planck equation associated to (1.6), (1.7) as well as numerical simulations. The authors of [14] claimed that the solutions of (1.6), (1.7) undergo inelastic collapse for $r < r_c$. More precisely, it was claimed that for $r < r_c$ the Brownian particle with inelastic collisions stops its motion in finite time with probability one, while for $r > r_c$ the Brownian particle with inelastic collisions has positive velocity for any time.

However, some papers in the physics literature raised doubts about the results in [14]. It was claimed in [17], [2], on the basis of numerical simulations, that "inelastic collapse" for $r < r_c$ cannot be observed.

There is a detailed discussion about these seemingly contradictory results in [8]. The conclusion of this paper was that for $r < r_c$ the particle $(X(t), V(t))$ solutions of (1.6), (1.7) arrive to $(x, v) = (0, 0)$ in finite time. However the particle $(X(t), V(t))$ does not stay there for later times. The main argument given in [8] to support this conclusion is a detailed analysis of a stationary solution of the Fokker-Planck equation associated to the problem (1.6), (1.7). Such stationary solution exhibits a flux of particles from the point $(x, v) = (0, 0)$ to the region $\{x > 0\}$. This flow of particles would be balanced by the flux of particles arriving towards the origin as predicted in [45] by McKean.

Suppose that we denote as $P(x, v, t)$ the probability density which gives the probability of finding one particle solving (1.6), (1.7) in any region of the phase space (x, v) . Inelastic collapse would imply that $P(x, v, t) \rightarrow \delta(x)\delta(v)$ as $t \rightarrow \infty$. On the other hand, the steady solution described in [8] is different from $\delta(x)\delta(v)$ and this suggests that the particles arriving to $(x, v) = (0, 0)$ do not necessarily remain there.

We will prove in this paper that if $0 < r < r_c$ the solutions of (1.3)-(1.5) are nonunique. More precisely, it is possible to obtain different nonnegative solutions of this Fokker-Planck equation with a different asymptotic behaviour as $(x, v) \rightarrow (0, 0)$. Actually, the solutions of (1.3)-(1.5) that we will construct in this paper will be measure-valued solutions and they will differ in the amount of mass that they contain at the point $\{(0, 0)\}$, i.e. $\int_{\{(0, 0)\}} P(dx dv, t)$.

The intuitive explanation behind this nonuniqueness result is the following. If $r < r_c$, the solutions of the problem (1.6), (1.7) reach the point $(0, 0)$ in a finite time with probability one. After reaching that point it is possible to give several dynamic laws for the evolution of the particle $(X(t), V(t))$. For instance, the particle could remain trapped at the origin, or it could continue their movement, or it could remain trapped at $(0, 0)$ during some time and then restart its motion again. The situation is essentially similar to the one of a Brownian particle moving in a half-line $\{x > 0\}$ which reaches $x = 0$ with probability one, and there, it can be either absorbed, or to be reflected and continue its motion back to the region $\{x > 0\}$. All these possibilities can be described in the case of this moving Brownian particle by means of different boundary conditions at $x = 0$ for the diffusion equation describing the probability density associated to that process. In the case of the Fokker-Planck equation (1.3)-(1.5) the different evolution laws assumed for $(X(t), V(t))$ after reaching the origin, will result in different boundary conditions for $P(x, v, t)$ as $(x, v) \rightarrow (0, 0)$.

The nonuniqueness of the stochastic process associated to (1.6), (1.7) has been considered in the literature of stochastic processes. The paper [6] describes the construction of a stochastic process which formally can be described by the stochastic differential equation $dX_t = V_t dt$, $dV_t = dW_t$, where W_t is the Wiener process, and where we impose in addition that $V_t = 0$ if $X_t = 0$, *a.s.* for $t \geq 0$. Moreover, the process constructed in [6] satisfies that the amount of time such that $X_t = 0$ has zero measure *a.s.* Notice that this stochastic process can be considered as a solution of (1.6), (1.7) with $r = 0$, except for the fact that the solution is not requested to remain in the half-plane $\{X_t \geq 0\}$. The surprising feature of the stochastic process obtained in [6] is the fact that "killing" the process at the times when X_t reaches the line $\{X_t = 0\}$ does not force the particle to stay at this line for later times. On the contrary, the particle is able to escape from the point $(X_t, V_t) = (0, 0)$ at later times. The results in [6] indicate that the solutions of (1.6), (1.7) can be expected to be nonunique.

On the other hand, the problem (1.6), (1.7) has been considered in [33], [34]. It has been shown in [33] that for $r \geq r_c$ there is a unique entrance law in the half plane $\{X > 0, V \in \mathbb{R}\}$ under the assumption that $X_0 = V_0 = 0$. On the other hand, if $r < r_c$ the bounces of the particle at the line $\{X = 0\}$ accumulate at the origin in a finite time. However, it is proved in [34] that the particle does not remain necessarily at $(X, V) = (0, 0)$ for later times, but on the contrary, it is possible to define a "resurrected" process after the particle reaches the origin. The measure of the times in which the solutions of this "resurrected" process remain in the line $\{X = 0\}$ is zero *a.s.*

In this paper we will obtain several results for the problem (1.3)-(1.5) which can be viewed as the PDE reformulation of the previously described results. More precisely, we will define a suitable concept of measured valued solutions for (1.3)-(1.5) if $r > r_c$. On the contrary it is possible to define different concepts of measured valued solutions P of (1.3)-(1.5) if $r < r_c$. The key point is that for $r < r_c$ we must impose some boundary condition at the singular point $(x, v) = (0, 0)$ in order to determine the solution of (1.3)-(1.5). Different boundary conditions can be thought as to be associated to different physical meanings for the corresponding evolution of the stochastic particle whose probability distribution is given by P . The boundary conditions obtained in this paper will be denoted as trapping, nontrapping and partially trapping boundary conditions at the point $(x, v) = (0, 0)$ and they can be given respectively the meaning of a particle which after reaching the point $(x, v) = (0, 0)$ remains trapped there for later times, alternatively does not remain trapped at $(x, v) = (0, 0)$ and

continuous its motion in the half plane $\{x > 0\}$ or remains trapped during some characteristic time and continues then its motion in the half-plane $\{x > 0\}$. We will not study in detail the case $r = r_c$ in this paper, because in that case some logarithmic terms appear in the asymptotics of the solutions near the singular point, something that makes the analysis rather cumbersome.

The different solutions P obtained will be nonnegative, classical solutions of (1.3)-(1.5) for $(x, v) \neq (0, 0)$ satisfying $\int_{\{(x,v) \neq (0,0)\}} P(x, v, t) dx dv < \infty$, but they differ in their asymptotics as $(x, v) \rightarrow (0, 0)$. These different solutions will satisfy different definitions of distributional solutions of (1.3)-(1.5) which will characterize the dynamics of probability densities $P(x, v, t)$ for which either the Brownian particles arriving to $(x, v) = (0, 0)$ remain trapped there, or they continue their motion or they remain trapped during a characteristic time before resuming their motion in the region $\{(x, v) \neq (0, 0), x > 0\}$. A more precise formulation of the results is contained in the following Theorem.

Theorem 1.1. *We will denote as \mathcal{X} the space of functions $C^\infty(\mathbb{R}^+ \times \mathbb{R} \times (0, \infty)) \cap C([0, \infty) : \mathcal{M}_+(\mathbb{R}^+ \times \mathbb{R})) \cap C(\overline{\mathbb{R}^+ \times \mathbb{R}} \times (0, \infty))$. Suppose that $0 < r < r_c$. For any Radon measure $P_0 \in \mathcal{M}_+(\mathbb{R}^+ \times \mathbb{R})$, such that $\int_{\mathbb{R}^+ \times \mathbb{R}} P_0 < \infty$, there exist infinitely many different solutions of the problem (1.3)-(1.5). The solutions $P = P(x, v, t) \in \mathcal{X}$ satisfy (1.3), (1.4) in classical sense and (1.5) in the sense of distributions. If $r_c < r \leq 1$ there exists a unique weak solution of (1.3)-(1.5) with initial data P_0 .*

Remark 1.2. *We will denote as $\mathcal{M}_+(B)$ the set of Radon measures in a given Borel set $B \subset \mathbb{R}^n$, $n \geq 1$.*

Precise definitions of solutions of (1.3)-(1.5) as well as more detailed information about the solutions described in Theorem 1.1 will be given later (cf. Chapter 28). Roughly speaking the solutions obtained for $0 < r < r_c$ differ in the boundary condition imposed at the singular point $(x, v) = (0, 0)$. Intuitively, in the case of the solution satisfying $\int_{\mathbb{R}^+ \times \mathbb{R}} P(\cdot, \cdot, t) = \int_{\mathbb{R}^+ \times \mathbb{R}} P_0$ we assume that the particles arriving to the origin continue instantaneously their motion. In all the other solutions, particles arriving to the singular point remain trapped there for all later times or during some characteristic time before resuming their motion in the region $\mathbb{R}^+ \times \mathbb{R}$. The existence and uniqueness of solutions for $r > r_c$ will be also proved in this paper, although in this case we do not need any additional information about the solutions near the singular set.

The main tool used in the proof of the results of this paper is the classical Hille-Yosida Theorem. Most of the technical difficulties in this paper arise from the fact that we need to prove that the operators involved in our problem satisfy the assumptions which allow to apply the Hille-Yosida Theorem. In particular, the application of this Theorem requires to prove the solvability of some problems of Partial Differential Equations which in particular encode information about the behaviour of the solutions of (1.3)-(1.5) near the singular set $\{(x, v) = (0, 0)\}$. Due to the fact that we are interested in Measured Valued solutions of (1.3)-(1.5) it will be convenient to work instead with the solutions of some suitable adjoint problems of (1.3)-(1.5) which encode the different types of boundary conditions mentioned above if $r < r_c$. In order to apply Hille-Yosida's Theorem we need to prove the solvability of some elliptic problems. This will be made by means of a suitable generalization of the classical Perron's method which is able to deal with the singular behaviour of the solutions near the singular point. It is well known that Perron's method allows us to obtain the solution of some elliptic equations as the supremum of the subsolutions associated to such equations. However, the application of such ideas to this problem yields to several technical difficulties. The main ones are the following ones. Since the operator $D_v^2 + vD_x$ contains

only the first order derivative in x instead of the second order ones, it is possible to construct subsolutions for this problem which are discontinuous and have discontinuities along subsets of lines $\{x = x_0\}$ for $x_0 \in \mathbb{R}$. Moreover, such discontinuous subsolutions arise naturally applying Perron's method to this class of equations. This fact will yield several technical difficulties which will be considered in Chapter 5. On the other hand the nonuniqueness of the solutions of (1.3)-(1.5) will be related to the different asymptotic behaviours of the solutions near the singular point. The application of Perron's method to the problem under consideration will require to study the properties of suitably defined sub and supersolutions with different behaviours near the singular point.

We will use repeatedly the asymptotic symbol \sim with the following meaning. Suppose that $f_1(x, v)$, $f_2(x, v)$ are two functions defined in some region \mathcal{D} whose boundary contains the origin $(x, v) = (0, 0)$. We will say that $f_1(x, v) \sim f_2(x, v)$ as $(x, v) \rightarrow (0, 0)$ if $\lim_{(x, v) \rightarrow (0, 0)} \frac{f_1(x, v)}{f_2(x, v)} = 1$ with $(x, v) \in \mathcal{D}$.

The plan of this paper is the following. In Chapter 2 we study a simple toy model, namely the random walks of particles in a one-dimensional half lattice. This yields a simple diffusion problem in the limit when the size of the lattice converges to zero for which it is easy to obtain the boundary conditions at the lower extreme of the lattice. The analysis of this simpler problem will make more clear the meaning of the boundary conditions imposed at the solutions of (1.3)-(1.5). Chapter 3 describes in heuristic form the different expected asymptotic behaviours of the solutions of (1.3), (1.4) near the singular point $(x, v) = (0, 0)$. These heuristic asymptotics are used to obtain, at the formal level, a set of adjoint problems to (1.3), (1.4) in Chapter 4. The onset of different adjoint problems is due to the possibility of having different asymptotics for the solutions of (1.3), (1.4) if $r < r_c$. Chapter 4 contains also a summary of hypoellipticity properties for the operator $D_v^2 + vD_x$ which are used in this paper. The well posedness of the adjoint problems, which have been precisely formulated in Chapter 4, is obtained in Chapter 5. This chapter is the most technical of the paper and it requires some detailed study of the asymptotics of the solutions of the adjoint problem near the singular point $(x, v) = (0, 0)$. The results of Chapter 5 are used in Chapter 6 to give a precise definition of measured valued solutions for the problem (1.3)-(1.5). Chapter 7 contains a study of the different types of stationary solutions of the problem (1.3), (1.4) in the strip $\{0 < x < 1, v \in \mathbb{R}\}$ depending on the different definitions of solutions given in Chapter 6. These stationary solutions are compared with the ones obtained in the physics literature (cf. [7], [8]). We will clarify in which sense they are distributional solutions of (1.3)-(1.5). In Chapter 8 we review the results which have been obtained for the problem (1.1) with the absorbing boundary conditions (1.2). Chapter 9 summarizes the results of this paper and it describes several open problems.

2. A TOY PROBLEM: DIFFUSION IN HALF-LINE.

In this section we discuss, using heuristic arguments, an elementary model which exhibits some similarities with the behaviour of the model (1.6), (1.7) in the case $r < r_c$. The results described in this Section are well known, since they just correspond to classical diffusion of a particle which reaches the boundary of a half-line. However, the results of this Section will help to clarify the nonuniqueness of solutions for (1.3)-(1.5) found in this paper and the physical meaning of the different types of solutions obtained for this problem.

Suppose that we consider the stochastic evolution of a particle in a lattice of points contained in the half-line $\{x \geq 0\}$. If the distance between the lattice points converges to zero, the probability density of finding the particle at one specific position can be approximated by means of the one-dimensional diffusion equation. However, different boundary conditions

must be assumed at $x = 0$ for this probability density depending on the type of dynamics prescribed for the particle which reach that point.

More precisely, for each $h > 0$, we will denote as \mathcal{L}_h the lattice:

$$\mathcal{L}_h = \{x_n = nh : n = 0, 1, 2, 3, \dots\}$$

For each $t_k = kh^2 : k = 0, 1, 2, \dots$, we will denote as $X(t_k)$ the position of one particle moving in the lattice \mathcal{L}_h with the following stochastic dynamics. If $X(t_k) = nh$, with $n > 1$ we will assign the following probability laws for the position of the particle at the time t_{k+1} :

$$p(X(t_{k+1}) = (n+1)h \mid X(t_k) = nh) = \frac{1}{2} \quad , \quad p(X(t_{k+1}) = (n-1)h \mid X(t_k) = nh) = \frac{1}{2}$$

On the other hand, we will define three different evolutions for the particle arriving to the point $x_0 = 0$. We will say that $X(\cdot)$ evolves with trapping boundary conditions at $x = 0$ if we assign the following probability:

$$(2.2) \quad p(X(t_{k+1}) = 0 \mid X(t_k) = 0) = 1$$

We will say that $X(\cdot)$ evolves with nontrapping boundary conditions at $x = 0$ if its evolution after reaching the point $x_0 = 0$ is given by:

$$(2.3) \quad p(X(t_{k+1}) = h \mid X(t_k) = 0) = \lambda \quad , \quad p(X(t_{k+1}) = 0 \mid X(t_k) = 0) = 1 - \lambda$$

where $\lambda \in (0, 1]$ independent of h . Notice that trapping boundary conditions are recovered from (2.3) if $\lambda = 0$.

Suppose that we write for short $P_n(k) = p(X(t_k) = nh)$. Then (2.1), (2.3) imply:

$$(2.4) \quad P_n(k+1) = \frac{1}{2} (P_{n-1}(k) + P_{n+1}(k)) \quad , \quad n \geq 2$$

$$(2.5) \quad P_1(k+1) = \frac{1}{2} P_2(k) + \lambda P_0(k) \quad , \quad P_0(k+1) = \frac{1}{2} P_1(k) + (1 - \lambda) P_0(k)$$

The structure of these equations suggests that in time scales where k is large (i.e. time scales $t \gg h^2$), the behaviour of the solutions of (2.3), (2.4) or (2.4), (2.5) should be given locally by steady states for all the values of n if $\lambda > 0$. Indeed, in this case, particles can leave the point $x_0 = 0$, and a local equilibrium can be expected. The local equilibria satisfy:

$$(2.6) \quad \varphi_n = \frac{1}{2} (\varphi_{n-1} + \varphi_{n+1}) \quad , \quad n \geq 2$$

$$(2.7) \quad \varphi_1 = \frac{1}{2} \varphi_2 + \lambda \varphi_0 \quad , \quad \varphi_0 = \frac{1}{2} \varphi_1 + (1 - \lambda) \varphi_0 \quad , \quad \lambda > 0$$

The solutions of (2.6) have the form $\varphi_n = A + Bn$, $n \geq 1$. Using (2.7) it then follows that

$$(2.8) \quad B = 0$$

If we define the family of measures $u_h(x, t) = \sum_{n,k} P_n(k) \delta_{x=x_n} \delta_{t=t_k}$ we would expect to have $u_h \rightharpoonup U$, where due to (2.4) it follows that:

$$(2.9) \quad \partial_t U = \frac{1}{2} \partial_{xx} U \quad , \quad x > 0 \quad , \quad t > 0$$

and where (2.8) yields the following boundary condition for U in the case of nontrapping boundary conditions:

$$(2.10) \quad \partial_x U(0, t) = 0$$

Suppose now that $\lambda = 0$. In this case, it does not make sense to assume that $P_0(k)$ is close to an equilibrium value, because any particle arriving to $x_0 = 0$ remains there for arbitrary times and therefore $P_0(\cdot)$ is increasing. Solving (2.6) we obtain $\varphi_n = A + Bn$, $n \geq 1$ and

using the first equation in (2.6) it then follows that $A = 0$. Therefore, in the case of trapping boundary conditions we obtain that U solves (2.9) with the boundary condition:

$$(2.11) \quad U(0, t) = 0$$

Notice that the solutions of (2.9), (2.11) do not preserve $\int_{(0, \infty)} U(x, t) dx$. This seems contradictory with the fact that $U(\cdot, t)$ is a probability density for all $t \geq 0$. However, the paradox is solved just taking into account that the mass lost from $(0, \infty)$ is transferred to the point $x = 0$. Therefore, the result to be expected in this case is:

$$(2.12) \quad u_h(x, t) \rightharpoonup U(x, t) + m(t) \delta_0, \quad m(t) = 1 - \int_{(0, \infty)} U(x, t) dx$$

It is possible to obtain an intermediate limit choosing λ in (2.5) depending on h . We will assume that:

$$(2.13) \quad \lambda = \mu h$$

for some $\mu > 0$, independent of h . The rationale behind this rescaling is the following. Since we are interested in obtaining U of order one in regions where x is of order one we can expect to have P_n of order h for $n \geq 1$. On the other hand, we are interested in the case in which there is a macroscopic fraction of mass at $x = 0$, whence P_0 is of order one. The second equation of (2.5) yields then $(P_0(k+1) - P_0(k)) = \frac{1}{2}P_1(k) - \lambda P_0(k)$. Given that the time scale between jumps is h^2 this suggests the approximation

$$(2.14) \quad [\partial_t m(t)] h^2 = \frac{1}{2}P_1(k) - \lambda P_0(k)$$

Plugging this into the first equation of (2.5) we would obtain

$$P_1(k+1) = \frac{1}{2}P_2(k) + \frac{1}{2}P_1(k) - [\partial_t m(t)] h^2$$

whence

$$P_1(k+1) - P_1(k) = \frac{1}{2}(P_2(k) - P_1(k)) - [\partial_t m(t)] h^2$$

Therefore, using that P_1, P_2 are of order h , the definition of u_h and the characteristic scales for the space and time jumps yield:

$$(2.15) \quad \frac{1}{2}\partial_x U(0, t) = \partial_t m(t)$$

This equation just provides the global mass conservation for the whole system. On the other hand, (2.14) yields, if $h^2 \ll \lambda$, that $\frac{1}{2}P_1(k) \approx \lambda P_0(k)$, whence using the scaling (2.13), as well as the fact that P_1 is of order h and P_0 is like $m(t)$:

$$(2.16) \quad m(t) = \frac{U(0, t)}{2\mu}$$

This boundary condition describes the equilibrium between the mass at $x = 0$ and the density probability in the surroundings. It is possible to reformulate it as a condition involving only local properties of the function U . Differentiating (2.16) with respect to t and using (2.15) we obtain:

$$(2.17) \quad U_t(0, t) = \mu U_x(0, t)$$

The problem (2.9), (2.17) describes the evolution of the probability density if λ rescales as in (2.13) in the limit $h \rightarrow 0$. The probability density, including the amount of the mass at the origin is given by (2.12). Notice that taking formally the limit $\mu \rightarrow \infty$ in (2.17) we recover the boundary condition (2.10). If we take $\mu = 0$ in (2.17) and we assume that $U(0, 0) = 0$, we recover the boundary condition (2.11).

The conclusion of these arguments is the following. If the underlying particle system described by the parabolic equation (2.9) yields a dynamics for which the particles can arrive to a boundary point of the domain we need to complete the equation with a suitable boundary condition. Different particle dynamics yield different boundary conditions. In particular, trapping boundary conditions yield (2.11) and nontrapping boundary conditions yield (2.10). Partially trapping boundary conditions, with an escape rate from $x = 0$ given by a coefficient λ , rescaling as (2.13) yield the boundary condition (2.17).

A way of describing the different types of boundary conditions in an unified way is the following. Suppose that we look for the following type of asymptotics for the solutions of (2.9):

$$(2.18) \quad U(x, t) \sim a(t) F(x) \quad \text{as } x \rightarrow 0$$

for a smooth function a and a suitable F which behaves algebraically as $x \rightarrow 0$. Notice that the asymptotics (2.18) does not make any assumption about boundary conditions for (2.9). Plugging (2.18) into (2.9) we would obtain that $\partial_{xx} F = 0$, whence we would have that, either F is proportional to 1 or to x . Then, the asymptotics of U near $x = 0$ can be expected to be given by:

$$(2.19) \quad U(x, t) \sim a_0(t) \cdot 1 + a_1(t) \cdot x + \dots \quad \text{as } x \rightarrow 0$$

The classical theory of boundary value problems for parabolic equations can be interpreted as stating that the meaningful boundary conditions for the solutions of (2.9) are those imposing a relation between $a_0(t)$ and $a_1(t)$. For instance, the homogeneous Dirichlet condition for (2.9) (cf. (2.11)) is equivalent to imposing $a_0(t) = 0$ in (2.19), while the homogeneous Neumann condition would mean $a_1(t) = 0$. The boundary condition (2.17) would yield $\frac{da_0}{dt} = \mu a_1$.

In the case of diffusion processes on the line, these results are well known. It is relevant to notice that the boundary conditions (2.10), (2.11), (2.17) can be thought as different asymptotics for U , near the boundary point under consideration. In the case of (1.3)-(1.5), different dynamical laws for the particles reaching the singular point $(x, v) = (0, 0)$ will result in different asymptotic formulas for $P(x, v, t)$ as $(x, v) \rightarrow (0, 0)$. There are, however, some technical difficulties arise due to the fact that the boundary conditions must be determined just at the point $(x, v) = (0, 0)$, although the problem is solved in the two-dimensional domain $\mathbb{R}^+ \times \mathbb{R}$ for $t > 0$.

3. HEURISTIC DESCRIPTION OF THE SOLUTIONS OF (1.3)-(1.5) NEAR THE SINGULAR POINT $(x, v) = (0, 0)$.

4. ASYMPTOTICS OF THE SOLUTIONS OF (1.3), (1.4) NEAR THE SINGULAR POINT.

In this Section we compute formally the asymptotic behaviour of the solutions of (1.3), (1.4) as $(x, v) \rightarrow (0, 0)$. This computation will illustrate the difference between the cases $r < r_c$ and $r > r_c$ and it will provide a simple intuitive explanation of the nonuniqueness arising for $r < r_c$. The insight gained in this Section will be useful later in order to define suitable concepts of weak solutions of (1.3)-(1.5).

Suppose that $P(x, v, t)$ is a solution of (1.3), (1.4) with the following asymptotic behaviour as $(x, v) \rightarrow (0, 0)$:

$$(4.1) \quad P(x, v, t) \sim a(t) G(x, v) \quad , \quad \text{as } (x, v) \rightarrow (0, 0)$$

where a is smooth and $G(x, v)$ has suitable homogeneity properties to be determined. Plugging (4.1) into (1.3) we would obtain the leading order terms

$$(4.2) \quad [\partial_t a(t)] G(x, v) \sim a(t) [-v \partial_x G(x, v) + \partial_{vv} G(x, v)]$$

Notice that, due to the presence of the derivatives with respect to x and v we can expect the terms on the left-hand side of (4.2) to be small compared with the ones on the right as $(x, v) \rightarrow (0, 0)$. It is then natural to assume, using also (1.4), that $G(x, v)$ satisfies:

$$(4.3) \quad v \partial_x G(x, v) = \partial_{vv} G(x, v) \quad , \quad x > 0, \quad v \in \mathbb{R},$$

$$(4.4) \quad G(0, v) = r^2 G(0, -rv) \quad , \quad v < 0.$$

This equation has been already found in the physics literature (cf. [8, 14]).

The invariance of the equations (4.3), (4.4) under the rescaling $x \rightarrow \lambda^3 x$, $v \rightarrow \lambda v$ suggests to look for solutions of these equations with the self-similar form:

$$(4.5) \quad G(x, v) = x^\gamma \Phi(z) \quad , \quad z = -\frac{v^3}{9x}$$

for some suitable γ to be determined. Before we give the result on the existence of such G , we start with the following lemma.

Lemma 4.1. *There exists a unique function $\alpha(r) : r \in \mathbb{R}_+ \rightarrow (-\frac{5}{6}, \frac{1}{6})$ such that*

$$(4.6) \quad (2 + 3\alpha(r)) \log r + \log \left(2 \cos \left(\pi \left(\alpha(r) + \frac{1}{3} \right) \right) \right) = 0.$$

Let

$$(4.7) \quad r_c = e^{-\frac{\pi}{\sqrt{3}}}$$

The function $r \rightarrow \alpha(r)$ is increasing in $r \in \mathbb{R}_+$. We have

$$\alpha(r_c) = -\frac{2}{3} \quad , \quad \alpha(1) = 0$$

and $\alpha(r) \in (-\frac{5}{6}, -\frac{2}{3})$ if $r < r_c$ and $\alpha(r) \in (-\frac{2}{3}, \frac{1}{6})$ if $r > r_c$. Moreover,

$$\lim_{r \rightarrow 0^+} \alpha(r) = -\frac{5}{6} \quad , \quad \lim_{r \rightarrow \infty} \alpha(r) = \frac{1}{6}$$

Proof. For any given $r \in \mathbb{R}_+$, consider

$$y_r(x) = (2 + 3x) \log r + \log \left(2 \cos \left(\pi \left(x + \frac{1}{3} \right) \right) \right).$$

Then, the functions $y_r(\cdot)$ are smooth for $x \in (-\frac{5}{6}, \frac{1}{6})$ and they satisfy $y_r(x) \rightarrow -\infty$ as $x \rightarrow -\frac{5}{6}$, $\frac{1}{6}$. Notice that $y_r(-\frac{2}{3}) = 0$ for all r . Then, the function $\bar{\alpha}(r) \equiv -\frac{2}{3}$ satisfies (4.6), but not the rest of conditions imposed to $\alpha(r)$ in Lemma 4.1. In order to show that there exists another solution of (4.6), we first show that the equation $y_r(x) = 0$ has a unique solution $x \in (-\frac{5}{6}, \frac{1}{6})$, $x \neq -\frac{2}{3}$ for $r \neq r_c$. Note that

$$\frac{dy_r}{dx} = 3 \log r - \pi \tan \left(\pi \left(x + \frac{1}{3} \right) \right).$$

Hence $\frac{dy_r}{dx} = 0$ has a unique solution $x_{rc} := \frac{1}{\pi} \arctan \left(\frac{3}{\pi} \log r \right) - \frac{1}{3}$ and $\frac{dy_r}{dx} > 0$ for $x < x_{rc}$ and $\frac{dy_r}{dx} < 0$ for $x > x_{rc}$ and thus $y_r(\cdot)$ has the maximum value at $x = x_{rc}$. Note that $x_{rc} \rightarrow -\frac{5}{6}$ as $r \rightarrow 0^+$; $x_{rc} \rightarrow \frac{1}{6}$ as $r \rightarrow \infty$; $x_{rc} = -\frac{2}{3}$ when $r = r_c$. Let $r < r_c$. Then $x_{rc} < -\frac{2}{3}$. Since $y_r(-\frac{2}{3}) = 0$, there exists a unique x_r so that $-\frac{5}{6} < x_r < x_{rc} < -\frac{2}{3}$ and $y(x_r) = 0$. Similarly, if $r > r_c$, there exists a unique x_r so that $-\frac{2}{3} < x_{rc} < x_r < \frac{1}{6}$ and $y(x_r) = 0$. Now if $r = r_c$, $x_{rc} = -\frac{2}{3}$ and hence the only solution of $y_r(\cdot) = 0$ is $x_r = -\frac{2}{3}$. Let us define $\alpha(r) := x_r$. Then it is now easy to deduce that $\alpha(r)$ satisfies all the properties in the Lemma. \square

We are now ready to state the result on the existence of G of the form (4.5) satisfying (4.3), (4.4).

Proposition 4.2. *For any $r > 0$, $r \neq r_c$, there are two linearly independent positive solutions of (4.3), (4.4) which take the form (4.5). They are analytic in the domain $\{(x, v) : x > 0, v \in \mathbb{R}\}$ and they have the form*

$$(4.8) \quad G_\gamma(x, v) = x^\gamma \Lambda_\gamma(\zeta) \quad , \quad \zeta = \frac{v}{(9x)^{\frac{1}{3}}} \quad \text{with } \gamma \in \left\{ -\frac{2}{3}, \alpha(r) \right\}$$

with

$$(4.9) \quad \Lambda_\gamma(\zeta) = U\left(-\gamma, \frac{2}{3}; -\zeta^3\right) > 0 \quad \text{for } \zeta \in \mathbb{R}$$

where we denote as $U(a, b; z)$ the classical Tricomi confluent hypergeometric functions (cf. [1]). Moreover, the asymptotic behaviour of $\Lambda_\gamma(\zeta)$, $\Lambda'_\gamma(\zeta)$ as $|\zeta| \rightarrow \infty$ is given by:

$$(4.10) \quad \Lambda_\gamma(\zeta) \sim \begin{cases} K_\gamma |\zeta|^{3\gamma}, & \zeta \rightarrow \infty, \\ |\zeta|^{3\gamma}, & \zeta \rightarrow -\infty. \end{cases}$$

$$(4.11) \quad \Lambda'_\gamma(\zeta) \sim \begin{cases} 3\gamma K_\gamma |\zeta|^{3\gamma-1}, & \zeta \rightarrow \infty, \\ -3\gamma |\zeta|^{3\gamma-1}, & \zeta \rightarrow -\infty. \end{cases}$$

where $\gamma \in (-\frac{5}{6}, \frac{1}{6})$ and $K_\gamma = 2 \cos(\pi(\gamma + \frac{1}{3}))$.

In order to prove Proposition 4.2 we will use the following Lemma:

Lemma 4.3. *For any $-5/6 < \gamma < 1/6$, we define:*

$$(4.12) \quad \Lambda_\gamma(\zeta) = U(-\gamma, \frac{2}{3}; -\zeta^3), \quad \zeta = \frac{v}{(9x)^{\frac{1}{3}}} \in \mathbb{R}.$$

with $U(a, b; z)$ as in Proposition 4.2. Then:

- (i) $G_\gamma(x, v) := x^\gamma \Lambda_\gamma(\zeta)$ satisfies (4.3).
- (ii) $\Lambda_\gamma(\zeta)$ is analytic in $\zeta \in \mathbb{C}$ and $\Lambda_\gamma(\zeta) > 0$ for any $\zeta \in \mathbb{R}$.
- (iii) The asymptotic behaviour of $\Lambda_\gamma(\zeta)$ for large $|\zeta|$, $\zeta \in \mathbb{R}$ is given by the formulas in (4.10).

Proof. The proof of (i) is just an elementary computation. We will show (ii) and (iii) are valid. In order to study the properties of $\Phi(z)$ for negative values of z we use that (cf. [1], 13.1.3):

$$(4.13) \quad U(a, b, z) = \frac{\pi}{\sin(\pi b)} \left(\frac{M(a, b, z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b} \frac{M(1+a-b, 2-b, z)}{\Gamma(a)\Gamma(2-b)} \right), \quad b \notin \mathbb{Z}.$$

The function $M(a, b, z)$ is analytic for all $z \in \mathbb{C}$. Combining (4.12) and (4.13) we obtain the following representation formula for $\Lambda_\gamma(\zeta)$:

$$(4.14) \quad \Lambda_\gamma(\zeta) = \frac{\pi}{\sin(\frac{2}{3}\pi)} \left(\frac{M(-\gamma, \frac{2}{3}, -\zeta^3)}{\Gamma(\frac{1}{3}-\gamma)\Gamma(\frac{2}{3})} + \zeta \frac{M(\frac{1}{3}-\gamma, \frac{4}{3}, -\zeta^3)}{\Gamma(-\gamma)\Gamma(\frac{4}{3})} \right), \quad \zeta \in \mathbb{R}.$$

Formula (4.14) provides a representation formula for $\Lambda(\zeta)$ in terms of the analytic functions $M(-\gamma, \frac{2}{3}, -\zeta^3)$, $M(\frac{1}{3}-\gamma, \frac{4}{3}, -\zeta^3)$. Therefore $\Lambda_\gamma(\zeta)$ is analytic in $\zeta \in \mathbb{C}$.

We can compute the asymptotics of $\Lambda_\gamma(\zeta)$ as $\zeta \rightarrow -\infty$ by using (4.12) and 13.5.2 in [1]. Then we deduce that

$$(4.15) \quad \Lambda_\gamma(\zeta) \sim |\zeta|^{3\gamma} \quad \text{as } \zeta \rightarrow -\infty.$$

On the other hand, the formula 13.5.1 in [1] yields the asymptotics:

$$(4.16) \quad M(a, b, z) \sim \frac{\Gamma(b) e^{i\pi a}}{\Gamma(b-a)} (z)^{-a}, \quad |z| \rightarrow \infty, \quad -\frac{\pi}{2} < \arg(z) < \frac{3\pi}{2}$$

In particular, choosing $z = re^{i\pi}$ we obtain:

$$(4.17) \quad M(a, b, -r) \sim \frac{\Gamma(b)}{\Gamma(b-a)} (r)^{-a}, \quad r \rightarrow \infty.$$

We remark that 13.5.1 in [1] gives also the asymptotic formula:

$$(4.18) \quad M(a, b, z) \sim \frac{\Gamma(b) e^{-i\pi a}}{\Gamma(b-a)} (z)^{-a}, \quad |z| \rightarrow \infty, \quad -\frac{3\pi}{2} < \arg(z) < -\frac{\pi}{2}$$

Notice that due to the analyticity of $M(a, b, z)$ in \mathbb{C} the asymptotic behaviour of $M(a, b, z)$ obtained along rectilinear paths approaching infinity and contained in $\{Re(z) < 0\}$ must be the same independently on which formula (4.16) or (4.18) is used. In particular, it is easy to see that (4.17) follows from (4.18) choosing $z = re^{-i\pi}$. Using (4.17) we obtain:

$$(4.19) \quad M(-\gamma, \frac{2}{3}, -\zeta^3) \sim \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{2}{3} + \gamma)} (\zeta)^{3\gamma} \text{ as } \zeta \rightarrow \infty$$

$$(4.20) \quad \zeta M(\frac{1}{3} - \gamma, \frac{4}{3}, -\zeta^3) \sim \frac{\Gamma(\frac{4}{3})}{\Gamma(1 + \gamma)} (\zeta)^{3\gamma} \text{ as } \zeta \rightarrow \infty$$

Combining (4.14), (4.19), (4.20) we obtain:

$$\Lambda_\gamma(\zeta) \sim \frac{\pi}{\sin(\frac{2}{3}\pi)} \left[\frac{1}{\Gamma(\frac{1}{3} - \gamma)} \frac{1}{\Gamma(\frac{2}{3} + \gamma)} + \frac{1}{\Gamma(-\gamma)\Gamma(1 + \gamma)} \right] (\zeta)^{3\gamma}, \quad \zeta \rightarrow \infty$$

Using then that $\Gamma(-x)\Gamma(1+x) = -\frac{\pi}{\sin(\pi x)}$ (cf. 6.1.17 in [1]) as well as the trigonometric formula $\frac{[\sin(\pi(\gamma + \frac{2}{3})) - \sin(\pi\gamma)]}{\sin(\frac{2}{3}\pi)} = 2 \cos(\pi(\gamma + \frac{1}{3}))$ we obtain

$$(4.21) \quad \Lambda_\gamma(\zeta) \sim K_\gamma |\zeta|^{3\gamma} \text{ as } \zeta \rightarrow \infty$$

with K_γ as in the statement of the Lemma. Notice that $K_\gamma > 0$ if $-5/6 < \gamma < 1/6$.

To finish the proof of Lemma 4.3, it only remains to prove that $\Lambda_\gamma(\zeta) > 0$ for any $\zeta \in \mathbb{R}$ and the considered range of values of γ . To this end, notice that if $\gamma \rightarrow 0$ we have $\Lambda_\gamma(\zeta) \rightarrow 1 > 0$ uniformly in compact sets of ζ . The functions $\Lambda_\gamma(\zeta)$ considered as functions of γ , change in a continuous manner. On the other hand, the asymptotic behaviors (4.15), (4.21) imply that the functions $\Lambda_\gamma(\zeta)$ are positive for large values of $|\zeta|$. If $\Lambda_\gamma(\cdot)$ has a zero at some $\zeta = \zeta_0 \in \mathbb{R}$ and $-5/6 < \gamma < 1/6$, then there should exist, by continuity, $-5/6 < \gamma_* < 1/6$ and $\zeta_* \in \mathbb{R}$ such that $\Lambda_{\gamma_*}(\zeta_*) = \frac{d}{d\zeta} \Lambda_{\gamma_*}(\zeta_*) = 0$. The uniqueness theorem for ODEs then implies that $\Lambda_{\gamma_*}(\cdot, \gamma_*) = 0$, but this would contradict the asymptotics (4.15), (4.21), whence the result follows. \square

Proof of Proposition 4.2. Assume G takes the form (4.5). Then Φ satisfies the following ODE:

$$(4.22) \quad z\Phi_{zz} + \left(\frac{2}{3} - z\right)\Phi_z + \gamma\Phi = 0.$$

It is also convenient to define the following auxiliary variable

$$(4.23) \quad \zeta = \frac{v}{(9x)^{\frac{1}{3}}}$$

so that $z = -\zeta^3$. Then $\Lambda_\gamma(\zeta) \equiv \Phi(-\zeta^3)$ satisfies the following ODE

$$(4.24) \quad \Lambda''_\gamma(\zeta) + 3\zeta^2\Lambda'_\gamma(\zeta) - 9\gamma\zeta\Lambda(\zeta) = 0.$$

For each γ , two independent solutions to (4.22) are given by the Kummer function $M(-\gamma, \frac{2}{3}; z)$ and Tricomi confluent hypergeometric function $U(-\gamma, \frac{2}{3}; z)$, (cf. [1]). In order to obtain a

solution of (4.22) which behaves algebraically as $z \rightarrow \pm\infty$, we recall the asymptotic behavior of $M(a, b; z)$ (see 13.1.4 and 13.1.5 in [1]):

$$(4.25) \quad \begin{aligned} M(a, b; z) &\sim \frac{\Gamma(b)}{\Gamma(a)} e^z z^{a-b} \quad \text{for } z \rightarrow \infty, \\ M(a, b; z) &\sim \frac{\Gamma(b)}{\Gamma(b-a)} (-z)^{-a} \quad \text{for } z \rightarrow -\infty. \end{aligned}$$

On the other hand, $U(-\gamma, \frac{2}{3}; z)$ behaves algebraically as $z \rightarrow \infty$ (see 13.5.2 in [1]). Therefore, in order to get the solutions satisfying the boundary condition (4.4), $\Phi(z)$ should be proportional to the Tricomi confluent hypergeometric function $U(-\gamma, \frac{2}{3}; z)$. Due to the linearity of the problem we can assume that the proportionality constant is one.

We now compute the behavior of $G_\gamma(x, v) = x^\gamma \Lambda_\gamma(\zeta)$ near the boundary. For $v < 0$ and $x \rightarrow 0^+$, since $\zeta \rightarrow -\infty$,

$$G_\gamma(x, v) = x^\gamma \Lambda_\gamma(\zeta) \sim x^\gamma \left| \frac{v}{(9x)^{\frac{1}{3}}} \right|^{3\gamma} = \frac{|v|^{3\gamma}}{9^\gamma}, \quad v < 0 \text{ and } x \rightarrow 0^+$$

and for $v > 0$ and $x \rightarrow 0^+$, since $\zeta \rightarrow \infty$,

$$G_\gamma(x, v) = x^\gamma \Lambda_\gamma(\zeta) \sim x^\gamma K_\gamma \left| \frac{v}{(9x)^{\frac{1}{3}}} \right|^{3\gamma} = K_\gamma \frac{|v|^{3\gamma}}{9^\gamma}, \quad v > 0 \text{ and } x \rightarrow 0^+$$

The boundary condition (4.4) implies that $\frac{|v|^{3\gamma}}{9^\gamma} = r^2 K_\gamma \frac{|rv|^{3\gamma}}{9^\gamma}$ whence $r^{2+3\gamma} K_\gamma = 1$. Therefore γ must satisfy the following

$$(4.26) \quad r^{2+3\gamma} \cdot 2 \cos \left(\pi \left(\gamma + \frac{1}{3} \right) \right) = 1.$$

Notice that $\gamma = -\frac{2}{3}$ always fulfills the condition (4.26) for all values of $r > 0$ and therefore $G_{-\frac{2}{3}}(x, v)$ is a solution of (4.3), (4.4) for all $r > 0$. The other value of γ satisfying (4.26) is given by $\alpha(r)$ from Lemma 4.1 since taking the logarithm of (4.26) yields (4.6). Therefore, $G_\gamma(x, v) = x^\gamma \Lambda_\gamma(\zeta)$, $\zeta = \frac{v}{(9x)^{\frac{1}{3}}}$ with $\gamma \in \{-\frac{2}{3}, \alpha(r)\}$ are two linearly independent positive solutions of (4.3), (4.4) for $r > 0$ and $r \neq r_c$. \square

Remark 4.4. We note that both functions $G_\gamma(x, v)$, $\gamma \in \{-\frac{2}{3}, \alpha(r)\}$ are integrable near the origin, i.e. $\int_{\{0 < x < 1, |v| < 1\}} G_\gamma(x, v) dx dv < \infty$. To this end we use the estimate $G_\gamma(x, v) \leq C x^\gamma$ if $|v| \leq x^{1/3}$ and $G_\gamma(x, v) \leq C |v|^{3\gamma}$ if $|v| > x^{1/3}$. Then, using the fact that $\gamma > -6/5$ we obtain:

$$\begin{aligned} \int_{\{0 < x < 1, |v| < 1\}} G_\gamma(x, v) dx dv &\leq C \int_{\{0 < x < 1, |v| < x^{1/3}\}} x^\gamma dx dv + C \int_{\{0 < x < |v|^3, |v| < 1\}} |v|^{3\gamma} dx dv \\ &\leq C \int_0^1 x^{\gamma+\frac{1}{3}} dx + C \int_{-1}^1 |v|^{3\gamma+3} dv \leq C \end{aligned}$$

Remark 4.5. We will denote from now on $\alpha(r)$ as α unless the dependence on r plays a role in the argument.

Remark 4.6. The asymptotics (4.10) is valid for arbitrary values of γ , although K_γ is not necessarily positive if γ is not contained in the interval $(-\frac{5}{6}, \frac{1}{6})$. We will occasionally use the asymptotics (4.10) with $\gamma = -\frac{2}{3}$.

We next evaluate the particle fluxes towards the origin associated to G_γ obtained in Proposition 4.2, which will be importantly used to characterize the boundary conditions at the singular point. We start with $G_{-\frac{2}{3}}(x, v)$.

We define a family of domains which will be repeatedly used in the following arguments.

Definition 4.7. *For any given $r > 0$ and any $b > 0$, we define:*

$$(4.27) \quad \mathcal{R}_{\delta,b} = \left\{ (x, v) : 0 \leq x^{\frac{1}{3}} \leq b^{\frac{1}{3}}\delta, -\delta \leq v \leq r\delta \right\}$$

We will denote as $\mathcal{R}_{\delta,1}$ the domain \mathcal{R}_δ .

Proposition 4.8. *Let $r > 0$ be given and $\mathcal{R}_{\delta,b}$ as in (4.27) for some $b > 0$. Then*

$$(4.28) \quad \int_{\partial\mathcal{R}_{\delta,b} \cap \{x>0\}} \left[-vG_{-\frac{2}{3}}n_x + \partial_v G_{-\frac{2}{3}}n_v \right] ds = 9^{\frac{2}{3}} \left[\log(r) + \frac{\pi}{\sqrt{3}} \right]$$

where $n = (n_x, n_v)$ is the unit normal vector to $\partial\mathcal{R}_{\delta,b}$ pointing towards $\mathcal{R}_{\delta,b}$.

We will use the following result in the proof of Proposition 4.8.

Lemma 4.9. $\Lambda(\zeta) = \Lambda_{-\frac{2}{3}}(\zeta)$ satisfies the following ODE

$$(4.29) \quad \Lambda'(\zeta) + 3\zeta^2\Lambda(\zeta) = 3$$

and moreover, $\Lambda(\zeta)$ is given by $\Lambda(\zeta) = 3 \int_{-\infty}^{\zeta} \exp(-\zeta^3 + s^3) ds$.

Proof. Recall the equation for Λ (4.24) when $\gamma = -2/3$:

$$\Lambda'' + 3\zeta^2\Lambda' + 6\zeta\Lambda = 0.$$

This equation is equivalent to $(\Lambda' + 3\zeta^2\Lambda)' = 0$ whence $\Lambda'(\zeta) + 3\zeta^2\Lambda(\zeta) = \Lambda'(0)$. Solving this equation we obtain:

$$(4.30) \quad \Lambda(\zeta) = C \exp(-\zeta^3) + \Lambda'(0) \int_{-\infty}^{\zeta} \exp(-\zeta^3 + s^3) ds$$

for some constant $C \in \mathbb{R}$. The function $\Lambda(\zeta)$ in (4.30) increases exponentially as $\zeta \rightarrow -\infty$ if $C \neq 0$. It then follows from (4.10) that $C = 0$, whence:

$$\Lambda(\zeta) = \Lambda'(0) \int_{-\infty}^{\zeta} \exp(-\zeta^3 + s^3) ds$$

Then, the asymptotics of $\Lambda(\zeta)$ as $\zeta \rightarrow -\infty$ is given by

$$\Lambda(\zeta) \sim \frac{\Lambda'(0)}{3\zeta^2} \quad \text{as } \zeta \rightarrow -\infty$$

whence (4.10) yields $\Lambda'(0) = 3$ and the result follows. \square

We will use also the following result.

Lemma 4.10. *Suppose that $\Lambda(\zeta)$ is as in Lemma 4.9. Then:*

$$(4.31) \quad \lim_{M \rightarrow \infty} \int_{-M}^M \zeta \Lambda(\zeta) d\zeta = \frac{\pi}{\sqrt{3}}.$$

Proof. Using the representation formula for $\Lambda(\zeta)$ obtained in Lemma 4.9 we obtain:

$$\ell = \lim_{M \rightarrow \infty} \int_{-M}^M \zeta \Lambda(\zeta) d\zeta = 3 \lim_{M \rightarrow \infty} \int_{-M}^M \left(\int_{-\infty}^{\zeta} \exp(-\zeta^3 + s^3) ds \right) \zeta d\zeta$$

Using the changes of variables $X = \zeta^3$, $Z = -s^3 + \zeta^3$ we obtain:

$$\ell = \frac{1}{3} \lim_{M \rightarrow \infty} \int_0^\infty \exp(-Z) dZ \int_{-M^3}^{M^3} \frac{dX}{X^{\frac{1}{3}}} \frac{1}{(X - Z)^{\frac{2}{3}}}$$

where it is understood in the following that the function $X \rightarrow X^{\frac{1}{3}}$ is defined for $X \in \mathbb{R}$ and it takes negative values for $X < 0$. In particular $(-X)^{\frac{1}{3}} = -X^{\frac{1}{3}}$ for $X \in \mathbb{R}$.

Replacing now the integration in X by integration in $t = \frac{X}{Z}$ we arrive at:

$$(4.32) \quad \ell = \frac{1}{3} \lim_{M \rightarrow \infty} \int_0^\infty \exp(-Z) \varphi\left(\frac{Z}{M}\right) dZ = \frac{1}{3} \lim_{M \rightarrow \infty} \int_0^\infty M \exp(-MZ) \varphi(Z) dZ$$

with:

$$(4.33) \quad \varphi(Z) = \int_{-\frac{1}{Z}}^{\frac{1}{Z}} \frac{dt}{t^{\frac{1}{3}}} \frac{1}{(t-1)^{\frac{2}{3}}}$$

Notice that if the limit $\varphi(0^+) = \lim_{Z \rightarrow 0^+} \varphi(Z)$ exists, with φ as in (4.33), it would follow from (4.32) that $\ell = \frac{\varphi(0^+)}{3}$. We then compute $\varphi(0^+)$ as follows. We first split the integral in (4.33) as $\varphi(Z) = \int_{-\frac{1}{Z}}^0 [\cdots] + \int_0^1 [\cdots] + \int_1^{\frac{1}{Z}} [\cdots]$. Using then the change of variables $t = -s + 1$ in the first integral and relabelling s as t we obtain:

$$(4.34) \quad \begin{aligned} \varphi(Z) &= - \int_1^{\frac{1}{Z}+1} \frac{dt}{(t-1)^{\frac{1}{3}}} \frac{1}{(t)^{\frac{2}{3}}} + \int_0^1 \frac{dt}{t^{\frac{1}{3}}} \frac{1}{(t-1)^{\frac{2}{3}}} + \int_1^{\frac{1}{Z}} \frac{dt}{t^{\frac{1}{3}}} \frac{1}{(t-1)^{\frac{2}{3}}} \\ &= \int_0^1 \frac{dt}{t^{\frac{1}{3}}} \frac{1}{(t-1)^{\frac{2}{3}}} - \int_{\frac{1}{Z}}^{\frac{1}{Z}+1} \frac{dt}{(t-1)^{\frac{1}{3}}} \frac{1}{(t)^{\frac{2}{3}}} + \\ &\quad + \int_1^{\frac{1}{Z}} \left[\frac{1}{t^{\frac{1}{3}}} \frac{1}{(t-1)^{\frac{2}{3}}} - \frac{1}{(t-1)^{\frac{1}{3}}} \frac{1}{(t)^{\frac{2}{3}}} \right] dt \end{aligned}$$

The first integral on the right-hand side of (4.34) can be computed using Beta functions. The second one can be estimated by $-\log(1-Z)$ and therefore it converges to zero as $Z \rightarrow 0^+$. The third integral on the right-hand side of (4.34) converges to an integral in $(1, \infty)$ as $Z \rightarrow 0^+$. Then, using that $B\left(\frac{2}{3}, \frac{1}{3}\right) = \frac{2}{3}\sqrt{3}\pi$ we obtain:

$$\varphi(0^+) = \frac{2}{3}\sqrt{3}\pi + \int_1^\infty \left[\frac{1}{t^{\frac{1}{3}}} \frac{1}{(t-1)^{\frac{2}{3}}} - \frac{1}{(t-1)^{\frac{1}{3}}} \frac{1}{(t)^{\frac{2}{3}}} \right] dt$$

The integral on the right can be transformed to a more convenient form using the change of variables $t = \frac{1}{x}$. Then:

$$(4.35) \quad \varphi(0^+) = \frac{2\sqrt{3}\pi}{3} + I,$$

where $I = \int_0^1 \frac{dx}{x} \left[\frac{1}{(1-x)^{\frac{2}{3}}} - \frac{1}{(1-x)^{\frac{1}{3}}} \right]$. We can compute I writing:

$$\begin{aligned} I &= \lim_{\varepsilon \rightarrow 0^+} \int_0^1 \frac{dx}{x^{1-\varepsilon}} \left[\frac{1}{(1-x)^{\frac{2}{3}}} - \frac{1}{(1-x)^{\frac{1}{3}}} \right] \\ &= \lim_{\varepsilon \rightarrow 0^+} \left[B\left(\varepsilon, \frac{1}{3}\right) - B\left(\varepsilon, \frac{2}{3}\right) \right] = \lim_{\varepsilon \rightarrow 0^+} \left(\Gamma(\varepsilon) \left(\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{1}{3} + \varepsilon)} - \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{2}{3} + \varepsilon)} \right) \right) \\ &= \left(\frac{\Gamma'(\frac{2}{3})}{\Gamma(\frac{2}{3})} - \frac{\Gamma'(\frac{1}{3})}{\Gamma(\frac{1}{3})} \right) \lim_{\varepsilon \rightarrow 0^+} (\varepsilon \Gamma(\varepsilon)) \end{aligned}$$

Using then 6.1.3 and 6.3.7 in [1], we obtain $I = \frac{\pi}{\sqrt{3}}$. Plugging this into (4.35) we obtain $\varphi(0^+) = \sqrt{3}\pi$. It then follows from (4.32) that $\ell = \frac{\pi}{\sqrt{3}}$ and therefore, the Lemma follows. \square

The previous Lemmas allow now to prove Proposition 4.8.

Proof of Proposition 4.8. We write

$$Q(\delta, b) = \int_{\partial \mathcal{R}_{\delta, b} \cap \{x > 0\}} \left[-v G_{-\frac{2}{3}} n_x + \partial_v G_{-\frac{2}{3}} n_v \right] ds.$$

The homogeneity of $G_{-\frac{2}{3}}$ and the definition of the domains $\mathcal{R}_{\delta, b}$ implies that $Q(\delta, b) = Q(1, b)$. On the other hand, Gauss Theorem and the differential equation (4.3) yield that $Q(1, b)$ is independent of b . Therefore, we just need to show that

$$(4.36) \quad \lim_{b \rightarrow 0^+} Q(1, b) = 9^{\frac{2}{3}} \left[\log(r) + \frac{\pi}{\sqrt{3}} \right]$$

Notice that the normal vector to $\partial \mathcal{R}_{1, b}$ is given by $n = (-1, 0)$ if $x = b$, $-1 \leq v \leq r$; $n = (0, -1)$ if $v = r$, $0 < x \leq b$, $n = (0, 1)$ if $v = -1$, $0 < x \leq b$. Therefore,

$$\begin{aligned} (4.37) \quad Q(1, b) &= - \int_0^b \partial_v G_{-\frac{2}{3}}(x, r) dx + \int_0^b \partial_v G_{-\frac{2}{3}}(x, -1) dx + \int_{-1}^r G_{-\frac{2}{3}}(b, v) v dv \\ &= (I_{11} + I_{12}) + I_2 = I_1 + I_2 \end{aligned}$$

We will compute I_1 : the first two integrals $I_{11} + I_{12}$. From (4.8), $\partial_v G_{-\frac{2}{3}}(x, v) = \frac{9^{-\frac{1}{3}}}{x} \Lambda'(\zeta)$ where $\Lambda \equiv \Lambda_{-\frac{2}{3}}$, and thus

$$(4.38) \quad I_1 = \frac{1}{9^{\frac{1}{3}}} \int_0^b \Lambda' \left(-\frac{1}{(9x)^{\frac{1}{3}}} \right) \frac{dx}{x} - \frac{1}{9^{\frac{1}{3}}} \int_0^b \Lambda' \left(\frac{r}{(9x)^{\frac{1}{3}}} \right) \frac{dx}{x}$$

Using the change of variables $x \rightarrow r^3 x$ in the last integral of (4.38) and splitting the resulting integral in the interval $(0, \frac{b}{r^3})$ in the integrals $\int_0^b [\dots] + \int_b^{\frac{b}{r^3}} [\dots]$ we obtain:

$$\begin{aligned} I_1 &= \frac{1}{9^{\frac{1}{3}}} \int_0^b \left[\Lambda' \left(-\frac{1}{(9x)^{\frac{1}{3}}} \right) - \Lambda' \left(\frac{1}{(9x)^{\frac{1}{3}}} \right) \right] \frac{dx}{x} - \frac{1}{9^{\frac{1}{3}}} \int_b^{\frac{b}{r^3}} \Lambda' \left(\frac{1}{(9x)^{\frac{1}{3}}} \right) \frac{dx}{x} \\ &= I_{1a} + I_{1b} \end{aligned}$$

Notice that I_{1b} depends on r whereas I_{1a} does not depend on r . We first estimate I_{1a} . Since $\Lambda(\zeta) = O(|\zeta|^{-2})$ as $|\zeta| \rightarrow \infty$, and due to the analyticity properties of this function from

Proposition 4.2, we have also $\Lambda'(\zeta) = O(|\zeta|^{-3})$ as $|\zeta| \rightarrow \infty$. It then follows that $\Lambda'\left(\frac{1}{(9x)^{\frac{1}{3}}}\right)$, $\Lambda'\left(-\frac{1}{(9x)^{\frac{1}{3}}}\right)$ are bounded for $0 < x \leq b \leq 1$, and

$$\left| \Lambda'\left(\frac{1}{(9x)^{\frac{1}{3}}}\right) \right| + \left| \Lambda'\left(-\frac{1}{(9x)^{\frac{1}{3}}}\right) \right| \leq Cx, \quad 0 < x \leq b \leq 1, \quad 0 < r \leq 1$$

which yields

$$(4.39) \quad |I_{1a}| \leq Cb,$$

where C is a uniform constant. Therefore

$$\lim_{b \rightarrow 0} I_{1a} = 0, \quad \lim_{b \rightarrow 0} Q(1, b) = \lim_{b \rightarrow 0} (I_{1b} + I_2).$$

In the rest of the Proof of this Proposition we compute this limit. Using (4.29) we can rewrite I_{1b} as:

$$(4.40) \quad \begin{aligned} I_{1b} &= -\frac{3}{9^{\frac{1}{3}}} \int_b^{\frac{b}{r^{\frac{1}{3}}}} \frac{dx}{x} + \frac{3}{9^{\frac{1}{3}}} \int_b^{\frac{b}{r^{\frac{1}{3}}}} \frac{1}{(9x)^{\frac{2}{3}}} \Lambda\left(\frac{1}{(9x)^{\frac{1}{3}}}\right) \frac{dx}{x} \\ &= 9^{\frac{2}{3}} \log(r) + 9^{\frac{2}{3}} \int_{r(9b)^{-\frac{1}{3}}}^{(9b)^{-\frac{1}{3}}} \Lambda(\zeta) \zeta d\zeta, \end{aligned}$$

where we have changed the variables to $\zeta = (9x)^{-\frac{1}{3}}$ for the second integral. On the other hand, using (4.8) we can write I_2 in (4.37) as $I_2 = \int_{-1}^r (b)^{-\frac{2}{3}} \Lambda\left(\frac{v}{(9b)^{\frac{1}{3}}}\right) v dv = 9^{\frac{2}{3}} \int_{-(9b)^{-\frac{1}{3}}}^{r(9b)^{-\frac{1}{3}}} \Lambda(\zeta) \zeta d\zeta$. Then:

$$\lim_{b \rightarrow 0} (I_{1b} + I_2) = 9^{\frac{2}{3}} \log(r) + 9^{\frac{2}{3}} \lim_{b \rightarrow 0} \int_{-(9b)^{-\frac{1}{3}}}^{r(9b)^{-\frac{1}{3}}} \Lambda(\zeta) \zeta d\zeta$$

and using Lemma 4.10 we obtain (4.36) and the Proposition follows. \square

Proposition 4.11. *Let $r > 0$ be given, $r \neq r_c$ with r_c as in (4.7). Let $\mathcal{R}_{\delta,b}$ be as in (4.27) with $b > 0$. Suppose that G_α is as in (4.8) with $\gamma = \alpha = \alpha(r)$, where $\alpha(r)$ is as in Lemma 4.1. Then:*

$$(4.41) \quad \lim_{\delta \rightarrow 0} \int_{\partial \mathcal{R}_{\delta,b} \cap \{x > 0\}} [-v G_\alpha n_x + \partial_v G_\alpha n_v] ds = 0$$

where $n = (n_x, n_v)$ is the unit normal vector to $\partial \mathcal{R}_{\delta,b}$ pointing towards $\mathcal{R}_{\delta,b}$.

Proof. We write $Q(\delta, b) = \int_{\partial \mathcal{R}_{\delta,b} \cap \{x > 0\}} [-v G_\alpha n_x + \partial_v G_\alpha n_v] ds$. Using Gauss Theorem we obtain that the function $Q(\delta, b)$ is independent of b , whence $Q(\delta, b) = Q(\delta, 1)$. On the other hand, using Gauss Theorem, as well as the boundary condition (4.4) we obtain that $Q(\delta, 1)$ is independent of δ . Moreover, the homogeneity of G_α and $\mathcal{R}_{\delta,b}$ imply that $Q(\delta, 1) = C\delta^{2+3\alpha}$, for some suitable constant $C \in \mathbb{R}$. The independence of $Q(\delta, 1)$ of δ then implies $C = 0$ and the result follows. \square

5. THE CASE $r < r_c$: TRAPPING, NONTRAPPING AND PARTIALLY TRAPPING BOUNDARY CONDITIONS.

The main heuristic idea behind the nonuniqueness results in this paper for $r < r_c$ as well as the role of the critical parameter r_c can be seen as follows. Proposition 4.2 suggests that an

integrable nonnegative solution of (1.3), (1.4) in $(x, v) \in \mathbb{R}_+ \times \mathbb{R}$ has the following asymptotic behaviour (cf. (4.1)):

$$(5.1) \quad P(x, v, t) \sim a_\alpha(t) G_\alpha(x, v) + a_{-\frac{2}{3}}(t) G_{-\frac{2}{3}}(x, v) \quad \text{as } (x, v) \rightarrow (0, 0)$$

for suitable functions $a_{-\frac{2}{3}}(t)$, $a_\alpha(t)$. Notice that for $r < r_c$ the most singular term in the right-hand side of (5.1) is $a_\alpha(t) G_\alpha(x, v)$ (assuming that $a_\alpha(t) \neq 0$). If we assume also that the functions $a_\alpha(t)$, $a_{-\frac{2}{3}}(t)$ are differentiable, we can expect to have corrective terms in (5.1) of order $(x + |v|^3)^{\beta + \frac{2}{3}}$, with $\beta = \min\{\alpha, -\frac{2}{3}\}$. Given that $\beta + \frac{2}{3} > \max\{\alpha, -\frac{2}{3}\}$ it then follows that such corrective terms would be negligible compared with the two terms on the right-hand side of (5.1).

By analogy with the one-dimensional diffusion process considered in Section 2 we can expect to be able to impose boundary conditions for P at the point $(x, v) = (0, 0)$ by means of a relationship between $a_{-\frac{2}{3}}(t)$ and $a_\alpha(t)$. In order to understand the meaning of those conditions we remark that the solution $a_{-\frac{2}{3}}(t) G_{-\frac{2}{3}}(x, v)$ is associated to particle fluxes towards $(x, v) = (0, 0)$ in the same manner as the term $a_1(t) \cdot x$ in (2.19) is associated to fluxes towards $x = 0$ for the solutions of (2.9). This can be seen by means of the following computation. Suppose that P has the asymptotics (5.1) and it decreases fast enough as $|(x, v)| \rightarrow \infty$, in order to avoid particle fluxes towards infinity. Let us denote as \mathcal{U} the set:

$$(5.2) \quad \mathcal{U} = \{(x, v) : x \geq 0, v \in \mathbb{R}, (x, v) \neq (0, 0)\}.$$

We will use also the following notation for the boundary of \mathcal{U} :

$$(5.3) \quad L^* = \{(0, v) : v \in \mathbb{R}\}$$

We compute $\partial_t \left(\int_{\mathcal{U}} P dx dv \right)$ using (1.3), (1.4):

$$(5.4) \quad \partial_t \left(\int_{\mathcal{U}} P dx dv \right) = \int_{\mathcal{U}} [-v \partial_x P + \partial_{vv} P] dx dv = \lim_{\delta \rightarrow 0} \int_{\mathcal{U} \setminus \mathcal{R}_\delta} [-v \partial_x P + \partial_{vv} P] dx dv$$

where \mathcal{R}_δ is as in Definition 4.7.

We can transform the integral on the right-hand side of (5.4) using Gauss Theorem. Then, the right-hand side of (5.4) can be transformed in:

$$(5.5) \quad \lim_{\delta \rightarrow 0} \int_{\{v < -\delta^3 \text{ or } v > r\delta^3\}} v P(0, v, t) dv + \lim_{\delta \rightarrow 0} \int_{\partial \mathcal{R}_\delta \cap \{x > 0\}} [-v P n_x + \partial_v P n_v] ds$$

where $n = (n_x, n_v)$ is the normal vector to $\partial \mathcal{R}_\delta$ away from \mathcal{R}_δ . The first integral in (5.5) vanishes due to (1.4). Using the asymptotics (5.1) we can then write the left-hand side of (5.4) as:

$$(5.6) \quad \begin{aligned} & a_{-\frac{2}{3}}(t) \lim_{\delta \rightarrow 0} \int_{\partial \mathcal{R}_\delta \cap \{x > 0\}} \left[-v G_{-\frac{2}{3}} n_x + \partial_v G_{-\frac{2}{3}} n_v \right] ds \\ & + a_\alpha(t) \lim_{\delta \rightarrow 0} \int_{\partial \mathcal{R}_\delta \cap \{x > 0\}} \left[-v G_\alpha n_x + \partial_v G_\alpha n_v \right] ds \end{aligned}$$

Using Propositions 4.8, 4.11 we can compute the limits in (5.6) whence:

$$(5.7) \quad \partial_t \left(\int_{\mathcal{U}} P dx dv \right) = -\kappa a_{-\frac{2}{3}}(t) \quad \text{with } \kappa = -9^{\frac{2}{3}} \left[\log(r) + \frac{\pi}{\sqrt{3}} \right].$$

We will now indicate how to define different types of boundary conditions for P , assuming that we have the asymptotics (5.1). Taking into account (5.7) it is natural to assume in the

case of nontrapping boundary conditions:

$$(5.8) \quad a_{-\frac{2}{3}}(t) = 0$$

Notice that in the asymptotics (5.1) the most singular term is $a_\alpha(t) G_\alpha(x, v)$. Since $G_\alpha(x, v) > 0$ (cf. Proposition 4.2), and $P \geq 0$ we must have $a_\alpha(t) \geq 0$.

On the other hand, arguing by analogy with the Toy model considered in Section 2, it is natural to define trapping boundary conditions by imposing that P is the smallest as possible. We will then define trapping boundary conditions by means of the condition:

$$(5.9) \quad a_\alpha(t) = 0$$

In principle, in order to show that the condition (5.9) is the one associated to trapping boundary conditions, one should study in detail the properties of a stochastic process in which particles can arrive to $(x, v) = (0, 0)$ in finite time and to impose that those particles remain there for later times. Alternatively, some heuristic justification of (5.9) by means of a discrete particle model, in the spirit of the Toy model considered in Section 2 could be given. We will not do neither of them in this paper. However, it is possible to provide some justification by means of PDE arguments of the fact that (5.9) is the condition that must be imposed in order to obtain a probability density $P(x, v, t)$ with trapping boundary conditions at $(x, v) = (0, 0)$. Indeed, notice that (5.1) and (5.9) as well as the fact that $P(x, v, t) \geq 0$ imply that $a_{-\frac{2}{3}}(t) \geq 0$. Using then (5.7) as well as the fact that $r < r_c$ it follows that for any nonnegative solution of (1.3), (1.4) satisfying (5.1), (5.9) we have $\partial_t \left(\int_{\mathcal{U}} P dx dv \right) \leq 0$, i.e. for these solutions the mass could be transferred from \mathcal{U} to $(x, v) = (0, 0)$, but not in the reverse way, as it would be expected for trapping boundary conditions. Moreover, for the class of solutions satisfying (5.1), the boundary condition (5.9) is necessary in order to have $\partial_t \left(\int_{\mathcal{U}} P dx dv \right) \leq 0$. Indeed, suppose that a positive solution of (1.3), (1.4) satisfies (5.1) with $a_\alpha(t) > 0$, $a_{-\frac{2}{3}}(t) < 0$ for some time interval. For such distributions we would have, due to (5.7) that $\partial_t \left(\int_{\mathcal{U}} P dx dv \right) > 0$. Therefore, (5.9) is the boundary condition that must be imposed for trapping boundary conditions if we assume that the solutions of (1.3), (1.4) satisfy (5.1).

By analogy with the boundary condition (2.16) for the Toy model in Section 2, we can look for boundary conditions in which a transfer of mass from $(x, v) = (0, 0)$ to the region of \mathcal{U} with $|(x, v)|$ small. Notice that since $\alpha < -\frac{2}{3}$ the measure measuring the fluxes of P , namely $-\partial_x P$ towards and from $\{x = 0\}$ with v small is very large. Therefore, this region can be expected to be close to equilibrium. Suppose that $m(t)$ is the total amount of mass of the measure P at $(x, v) = (0, 0)$. Since P is described to the leading order in \mathcal{U} , with (x, v) close to $(0, 0)$ by means of $a_\alpha(t) G_\alpha(x, v)$. Due to the linearity of the problem, the local equilibrium between the mass of P at $(0, 0)$ and its surroundings requires a condition with the form:

$$(5.10) \quad a_\alpha(t) = \mu_* m(t)$$

for some $\mu_* \geq 0$. The nonnegativity of μ_* is due to the fact that $a_\alpha(t) \geq 0$, $m(t) \geq 0$. Notice that in the case $\mu_* = 0$ (5.10) reduces to the trapping boundary condition (5.9).

Differentiating (5.10), and using that, due to (5.7) we have

$$\frac{dm}{dt} = -9^{\frac{2}{3}} \left[\log(r) + \frac{\pi}{\sqrt{3}} \right] a_{-\frac{2}{3}}(t),$$

and we obtain the following boundary condition

$$(5.11) \quad \frac{da_\alpha}{dt} = -9^{\frac{2}{3}} \left[\log(r) + \frac{\pi}{\sqrt{3}} \right] \mu_* a_{-\frac{2}{3}}$$

We can interpret the solutions of (1.3), (1.4), (5.11) as the probability density associated to a particle system with the property that a particle reaching the point $(x, v) = (0, 0)$ returns to \mathcal{U} to the rate μ_* . Notice that in the limit case $\mu_* = \infty$ (5.11) formally reduces to the nontrapping boundary condition (5.8).

6. THE CASE $r > r_c$: NONTRAPPING BOUNDARY CONDITIONS AND PARTICLE FLUXES FROM $(0, 0)$ TO \mathcal{U} .

In the case $r > r_c$ (including $r \geq 1$), the situation changes completely with respect to the case $r < r_c$ due to the fact that we now have $\alpha = \alpha(r) > -\frac{2}{3}$, and also because the right-hand side of (4.28) becomes now nonnegative.

Suppose that the asymptotics (5.1) holds. Then, to the leading order P can be approximated near $(x, v) = (0, 0)$ by means of $a_{-\frac{2}{3}}(t) G_{-\frac{2}{3}}(x, v)$. The nonnegativity of P implies that $a_{-\frac{2}{3}}(t) \geq 0$, and therefore (5.7) yields $\partial_t \left(\int_{\mathcal{U}} P dx dv \right) \geq 0$. Moreover, if $a_{-\frac{2}{3}}(t) > 0$ we would obtain $\partial_t \left(\int_{\mathcal{U}} P dx dv \right) > 0$. Therefore, if $r > r_c$ the only boundary condition which is compatible with solutions defining a probability measure in $\mathcal{U} \cup \{(0, 0)\}$ for general initial data is the nontrapping boundary condition, namely:

$$(6.1) \quad a_{-\frac{2}{3}}(t) = 0$$

Notice that the previous argument does not imply that it is impossible to construct solutions of the PDE problem (1.3), (1.4) satisfying (5.1) and having $a_{-\frac{2}{3}}(t) \neq 0$. The problem is that those solutions cannot be understood in general as probability distributions. Indeed, if $a_{-\frac{2}{3}}(t) < 0$ we would have $P(x, v, t) < 0$ for $(x, v) \in \mathcal{U}$ small, and then, the resulting function P would not be a probability density. On the other hand it is possible to have positive solutions of (1.3), (1.4) satisfying (5.1) with $a_{-\frac{2}{3}}(t) > 0$. However, for such solutions $\int_{\mathcal{U}} P dx dv$ is an increasing function of t . It would be possible to obtain conserved measures $m\delta_{(0,0)} + P$ assuming that initially $m > 0$ and $m + \int_{\mathcal{U}} P dx dv = 1$, and having $a_{-\frac{2}{3}}(t) > 0$ only during the range of times in which $\int_{\mathcal{U}} P dx dv < 0$. Since those boundary conditions do not allow to interpret P as a probability density they will not be considered in this paper, although they could be useful in some problems. We could also have $P > 0$, $a_{-\frac{2}{3}}(t) > 0$ if we do not impose that P is the restriction of a probability measure to \mathcal{U} but, say, a particle density. The corresponding solutions would represent then, particle densities for which there is a flux of particles from $(x, v) = (0, 0)$ to \mathcal{U} . It is interesting to remark that the same fluxes from $(x, v) = (0, 0)$ to \mathcal{U} can be obtained if $r < r_c$, although in that case, we must assume that $a_{-\frac{2}{3}}(t) < 0$ in order to obtain an increasing number of particles in \mathcal{U} . Nevertheless, we will restrict our attention in the following just to the solutions of (1.3), (1.4) satisfying (5.1) as well as one of the boundary conditions (5.8), (5.9), (5.11) if $r < r_c$ and (6.1) if $r > r_c$. Indeed, for those boundary conditions we have natural interpretations for P as the probability density describing the evolution of a particle with nontrapping, trapping or partially trapping boundary conditions.

7. DEFINITION OF A PROBABILITY MEASURE IN $\mathcal{V} = \overline{\mathcal{U}}$.

Suppose that P is a solution of (1.3)-(1.5) in \mathcal{U} satisfying (5.1) as well as one of the boundary conditions (5.9), (5.11), (5.8) (if $r < r_c$) or (6.1) (if $r > r_c$). In order to have mass conservation, it is convenient to define a probability measure f in the domain $\mathcal{V} = \overline{\mathcal{U}}$ as follows:

$$(7.1) \quad f(x, v, t) = P(x, v, t) + m(t) \delta_{(0,0)}(x, v)$$

where:

$$(7.2) \quad \int_{\mathcal{U}} P(\cdot, t) dx dv + m(t) = 1$$

for $t \geq 0$.

Notice that due to (7.2) the measure f is a probability measure. In the case $r > r_c$, where the boundary condition which we need to impose is (6.1), or in the case $r < r_c$, if we assume the boundary condition (5.8) we have that $\int_{\mathcal{U}} P(\cdot, t) dx dv$ is constant. If we assume that $\int_{\mathcal{U}} P_0(\cdot) dx dv = 1$, it then follows that in those cases $m(t) = 0$, $t \geq 0$.

Notice that in order to define the measure f we do not need to have (5.1). Since we will use later functions P for which the detailed asymptotics (5.1) will not be rigorously proved, we remark that the definition (7.1), (7.2) is meaningful if we have, say $P \in L^1((0, T) : L^1(\mathcal{U}))$ for $0 < T \leq \infty$. In particular the detailed boundary conditions (5.8), (5.9), (5.11) or (6.1) are not needed. We can formulate this more precisely as follows.

Definition 7.1. *Given $P \in L^1((0, T) : L^1(\mathcal{U}))$ for $0 < T \leq \infty$ we define $f \in L^1((0, T) : \mathcal{M}(\mathcal{V}))$ by means of (7.1) with $m(t)$ as in (7.2) for a.e. $t \in (0, T)$.*

8. FORMULATION OF THE ADJOINT PROBLEMS.

Our goal is to construct solutions of (1.3)-(1.5) satisfying (5.1) as well as one of the boundary conditions (5.9), (5.11), (5.8) (if $r < r_c$) or (6.1) (if $r > r_c$). However, we will not attempt to derive asymptotic formulas as detailed as (5.1). Instead, we will find a weak formulation of the boundary conditions (5.9), (5.11), (5.8), (6.1). In order to do this we will define suitable adjoint problems. These adjoint problems will have some advantages over the original ones, in particular its solutions will be bounded, in contrast with the asymptotics (5.1). On the other hand the adjoint problems can be studied in a natural way using Hille-Yosida Theorem. After proving the well-posedness of these adjoint problems we will obtain a natural definition of measured valued solutions of (1.3)-(1.5) satisfying the conditions (5.9), (5.11), (5.8), (6.1) in a weak form. Therefore, we then begin deriving formally the adjoint problem of (1.3), (1.4) for each of the boundary conditions (5.9), (5.11), (5.8), (6.1).

Definition 8.1. *Suppose that P is a smooth function in \mathcal{U} which satisfies (5.1) and it solves (1.3), (1.4), with one of the boundary conditions (5.8), (5.9), (5.11), (6.1). We will say that the operator \mathbb{A} defined in a set of functions $\mathcal{D}(\mathbb{A}) \subset C^2(\mathbb{R}_+^2)$ is the adjoint of the evolution given by (1.3), (1.4) and the corresponding boundary condition if for any function smooth outside the origin P solving (1.3), (1.4) with the corresponding boundary condition, and for functions $\varphi = \varphi(x, v, t)$, with $\left(|x| + |v|^3\right)^\alpha |\partial_t \varphi|$, $\left(|x| + |v|^3\right)^\alpha |\mathbb{A}(\varphi)| \in L^1([0, T] \times \mathbb{R}_+^2)$, $\varphi(\cdot, t) \in \mathcal{D}(\mathbb{A})$ for any $t \in [0, T]$, the identity:*

$$(8.1) \quad \int_{\mathcal{V}} \varphi(x, v, T) f(dx dv, T) - \int_{\mathcal{V}} \varphi(x, v, 0) f(dx dv, 0) = \int_0^T \int_{\mathcal{U}} P(\varphi_t + \mathbb{A}(\varphi)) dx dv dt$$

holds, where f is as in Definition 7.1.

Remark 8.2. *We will say that P is smooth if P_{vv} , P_x , P_t exist and are continuous in $\mathcal{U} \times (0, T)$.*

Remark 8.3. *Notice that (8.1) implies the following. If φ satisfies the conditions in Definition 8.1 and, in addition, it has the property that $\varphi_t + \mathbb{A}(\varphi) = 0$ in \mathcal{U} , then:*

$$\int_{\mathcal{V}} \varphi(x, v, T) f(dx dv, T) = \int_{\mathcal{V}} \varphi(x, v, 0) f(dx dv, 0)$$

Remark 8.4. Notice that the adjoint operators \mathbb{A} defined in Definition 8.1 are not necessarily unique, because we could consider different domains $\mathcal{D}(\mathbb{A})$. It would be possible to obtain uniqueness of the adjoint operator assuming that the elements of $\mathcal{D}(\mathbb{A})$ are in suitable functional spaces. However this would complicate the notation in an unnecessary way and it will not be needed in the following arguments.

In order to derive the form of the operator \mathbb{A} we need to assume that the class of solutions of the problem (1.3), (1.4), with any of the boundary conditions (5.8), (5.9), (5.11), (6.1), is large enough. This will be formalized in the following property which we state here for further reference.

Definition 8.5. We denote as \mathcal{S} the set of smooth solutions of any of the problems (1.3), (1.4) with one of the boundary conditions (5.8), (5.9), (5.11), (6.1) in an interval $t \in [0, T]$. We will say that the set \mathcal{S} has the Property (P) if the identity:

$$\int_{\mathcal{U} \times (0, T)} Ph dx dv dt = 0 \quad \text{for any } P \in \mathcal{S}$$

implies $h = 0$ in $\mathcal{U} \times (0, T)$ and

$$\int_{\{x=0, v>0\} \times (0, T)} h P dv dt = 0 \quad \text{for any } P \in \mathcal{S}$$

implies $h = 0$ in $\{x = 0, v > 0\} \times (0, T)$.

Remark 8.6. We will not prove in this paper that the solutions of the problem (1.3), (1.4) with any of the boundary conditions (5.8), (5.9), (5.11), (6.1) has the Property (P) above. We will only use this definition to derive the form of the adjoint operators \mathbb{A} in Definition 8.1. We just remark that the Property (P) is a natural assumption for any equation having a set of solutions sufficiently large.

Remark 8.7. Smooth solutions in Definition 8.5 just means that all the derivatives appearing in (1.3) exist and are continuous functions in \mathcal{U} .

9. DERIVATION OF THE ADJOINT EQUATION AND THE BOUNDARY CONDITIONS AWAY FROM THE SINGULAR POINT.

It turns out that the adjoint operators \mathbb{A} defined in Definition 8.1 are given by a second order differential operator in \mathcal{U} , for functions φ such that $\text{supp}(\varphi(\cdot, t)) \cap \{(0, 0)\} = \emptyset$ for $t \in [0, T]$. Moreover, we can obtain also a set of boundary conditions for the functions in $\mathcal{D}(\mathbb{A})$ at the boundaries $L^* \times (0, \infty) = \{(x, v, t) = (0, v, t) : v \in \mathbb{R}, t > 0\}$. It is worth to remark that the action of the operators \mathbb{A} , in functions φ with $\text{supp}(\varphi) \cap \{(0, 0)\} = \emptyset$ as well as the corresponding boundary conditions, are the same for all the set of boundary conditions (5.8), (5.9), (5.11), (6.1).

Proposition 9.1. Suppose that \mathbb{A} is the adjoint of the evolution (1.3), (1.4) with any of the boundary conditions (5.8), (5.9), (5.11), (6.1) in the sense of Definition 8.1. Suppose that for any $t \in [0, T]$ we have $\varphi(\cdot, t) \in C^2(\mathcal{V})$, with $\text{supp}(\varphi(\cdot, t)) \cap \{(0, 0)\} = \emptyset$ for any $t \in [0, T]$ and that $|\partial_t \varphi|$, $|\mathbb{A}(\varphi)|$ satisfy the integrability conditions in Definition 8.1. Suppose that the set of solutions \mathcal{S} defined in Definition 8.5 satisfies Property (P). Then we have:

$$(9.1) \quad \mathbb{A}\varphi(x, v, t) = v \partial_x \varphi(x, v, t) + \partial_{vv} \varphi(x, v, t)$$

$$(9.2) \quad \varphi(0, rv, t) = \varphi(0, -v, t) \quad , \quad v > 0 \quad , \quad t > 0$$

Proof. Suppose that $\varphi = \varphi(x, v, t)$ is a test function whose support is contained in $(x, v, t) \in \mathcal{U} \times (0, \infty)$. Multiplying (1.3) by φ and integrating by parts in $\mathcal{U} \times (0, \infty)$ we obtain:

$$(9.3) \quad \int_{\mathcal{U} \times (0, \infty)} P(-\partial_t \varphi - v \partial_x \varphi - \partial_{vv} \varphi) dx dv dt + \int_{L^* \times (0, \infty)} v \varphi P dv dt = 0$$

Suppose first that the support of φ does not intersect $L^* \times (0, \infty)$. Since P is an arbitrary solution of (1.3), (1.4) with one of the conditions (5.8), (5.9), (5.11), (6.1) it then follows that:

$$(9.4) \quad \partial_t \varphi(x, v, t) + v \partial_x \varphi(x, v, t) + \partial_{vv} \varphi(x, v, t) = 0 \quad , \quad x > 0, \quad v \in \mathbb{R} \quad , \quad t > 0$$

It then follows from (1.4) and (9.3) that:

$$\int_0^\infty dt \int_0^\infty dv \left[\varphi(0, v, t) - \varphi\left(0, -\frac{v}{r}, t\right) \right] v P(0, v, t) = 0$$

Since this identity holds for arbitrary solutions of (1.3), (1.4) we can use again Property (P) to obtain (9.2). \square

The problem (9.4), (9.2) defines the adjoint problem of (1.3), (1.4) for test functions φ whose support does not contain the singular point. However, the problem (9.4), (9.2) does not define uniquely an evolution semigroup if $r < r_c$ and additional conditions concerning the asymptotics of φ as $(x, v) \rightarrow (0, 0)$ are required in order to prescribe uniquely an evolution problem for φ . In order to determine this set of boundary conditions we first study the possible asymptotics of the solutions of (9.4), (9.2) near the singular point. The arguments used to derive the asymptotics of (9.4), (9.2) will be formal, close in spirit to those yielding (5.1). Rigorous asymptotic expansions for the functions φ will be made precise and obtained later.

10. ASYMPTOTICS OF THE SOLUTIONS OF (9.4), (9.2) NEAR THE SINGULAR POINT.

We compute formally the possible asymptotic behaviour of the solutions of (9.4), (9.2) with a method similar to the one used in Section 3 to compute the asymptotics of the solutions of (1.3), (1.4). More precisely, we look for solutions of (9.4), (9.2) with the form:

$$(10.1) \quad \varphi(x, v, t) \sim a(t) F(x, v) \quad \text{as} \quad (x, v) \rightarrow (0, 0)$$

where F behaves algebraically near the singular point and $a(t)$ is a smooth function. Arguing as in Section 3 it then follows that F must be a solution of the stationary problem:

$$(10.2) \quad v \partial_x F + \partial_{vv} F = 0, \quad x > 0, \quad v \in \mathbb{R},$$

$$(10.3) \quad F(0, rv) = F(0, -v), \quad v > 0.$$

The invariance of (10.2), (10.3) under the rescaling $x \rightarrow \lambda^3 x$, $v \rightarrow \lambda v$ suggests to look for solutions of this problem with the form:

$$(10.4) \quad F_\beta(x, v) = x^\beta \Phi(y) \quad , \quad y = \frac{v^3}{9x}$$

Lemma 10.1. *There exists a function $\beta(r) : r \in \mathbb{R}_+ \rightarrow (-\frac{5}{6}, \frac{1}{6})$ such that*

$$(10.5) \quad -3\beta(r) \log(r) + \log\left(2 \sin\left(\pi\left(\frac{1}{6} - \beta(r)\right)\right)\right) = 0 \quad , \quad \beta(r) \in \left(-\frac{5}{6}, \frac{1}{6}\right)$$

We have:

$$\beta(r_c) = 0 \quad , \quad \beta(1) = -\frac{2}{3}$$

with r_c as in (4.7). Moreover:

$$\lim_{r \rightarrow 0^+} \beta(r) = \frac{1}{6} \quad , \quad \lim_{r \rightarrow \infty} \beta(r) = -\frac{5}{6}$$

The function $\beta(r)$ is related with the function $\alpha(r)$ obtained in Proposition 4.1 by means of:

$$(10.6) \quad \beta(r) = -\alpha(r) - \frac{2}{3}$$

Proof. The equation (10.5) can be transformed into (4.6) by means of the change of variables (10.6). The result then follows from Lemma 4.1. \square

Proposition 10.2. *For any $r > 0$ the function $F_0(x, v) = 1$ is a solution of (10.2), (10.3). Moreover, for any $r > 0$, $r \neq r_c$, there exists another linearly independent positive solution F_β with the form (10.4) with $\beta = \beta(r)$ as in Lemma 10.1. The function F_β is analytic in $\{(x, v) : x > 0, v \in \mathbb{R}\}$ and it has the form:*

$$(10.7) \quad F_\beta(x, v) = x^\beta \Phi_\beta(y) \quad \text{with} \quad \Phi_\beta(y) = U\left(-\beta, \frac{2}{3}; y\right) \quad , \quad y = \frac{v^3}{9x}$$

The asymptotics of $\Phi_\beta(y)$ is given by:

$$(10.8) \quad \Phi_\beta(y) \sim |y|^\beta \quad \text{as } y \rightarrow \infty,$$

$$(10.9) \quad \Phi_\beta(y) \sim K |y|^\beta \quad \text{as } y \rightarrow -\infty$$

where $K = r^{3\beta}$.

Proof. We look for solutions of (10.2), (10.3) with the form (10.4). Then Φ satisfies:

$$y\Phi_{yy} + \left(\frac{2}{3} - y\right)\Phi_y + \beta\Phi = 0.$$

Notice that this equation is the same as (4.22). The solution of this equation yielding algebraic behaviour as $|y| \rightarrow \infty$ and satisfying the normalization (10.8) is $\Phi_\beta(y) = U(-\beta, \frac{2}{3}; y)$. The asymptotics (10.9) with $K = r^{3\beta}$ follows from Proposition 4.2. It then follows from these asymptotic formulas combined with the fact that $F_\beta(x, v) = x^\beta \Phi_\beta(y)$ that $F_\beta(0^+, v) = \frac{1}{9^\beta} v^{3\beta}$ for $v > 0$ and $F_\beta(0^+, v) = \frac{K}{9^\beta} |v|^{3\beta}$ for $v < 0$, whence the boundary condition (10.3) follows using the value of K . \square

Notice that (10.1) and Lemma 10.1 suggest the following asymptotics for the function φ near the singular point:

$$(10.10) \quad \varphi(x, v, t) \sim b_0(t) F_0(x, v) + b_\beta(t) F_\beta(x, v) + \dots \quad \text{as } (x, v) \rightarrow (0, 0)$$

Using the formal asymptotics (5.1), (10.10) we can obtain precise formulations for the adjoint problems of the problems defined by means of (1.3), (1.4) with one of the boundary conditions (5.8), (5.9), (5.11), (6.1). More precisely, we will encode the asymptotics (10.10) in the domains of the operators \mathbb{A} in Definition 8.1. Our next goal is to define several operators Ω_σ depending on the boundary conditions under consideration, which will be proved to be the adjoints \mathbb{A} defined in Definition 8.1 for the different sets of boundary conditions under consideration. We first compute an integral which will be used in the derivation of some of the adjoint operators. This computation is a bit tedious and technical, although it just uses classical tools of Complex Analysis, like contour deformations and the computation of suitable limits.

10.1. Computation of an integral related to particle fluxes. We will need to compute the following limit:

$$(10.11) \quad C_* = \lim_{\delta \rightarrow 0} \int_{\partial \mathcal{R}_\delta} [G_\alpha (n_v D_v F_\beta + n_x v F_\beta) - F_\beta D_v G_\alpha n_v] ds$$

where \mathcal{R}_δ is as in (4.27) and $n = (n_x, n_v)$ is the normal vector to $\partial \mathcal{R}_\delta$ pointing towards \mathcal{R}_δ . We have the following result.

Proposition 10.3. *Suppose that $0 < r < r_c$ and \mathcal{R}_δ is as in (4.27). Let us assume also that α is as in Lemma 4.1, β as in Lemma 10.1, G_α is defined as in (4.8) with $\gamma = \alpha$, F_β as in Proposition 10.2 and n is the normal vector to $\partial \mathcal{R}_\delta$ pointing towards \mathcal{R}_δ . Then the constant C_* defined in (10.11) takes the following value:*

$$(10.12) \quad \frac{C_*}{9^{\frac{2}{3}}} = \frac{\pi}{3} \left(\sin(\pi\alpha) + \sqrt{3} \cos(\pi\alpha) \right) - 2 \cos \left(\pi \left(\beta + \frac{1}{3} \right) \right) \log(r)$$

We will use that $C_* < 0$ for $0 < r < r_c$.

Lemma 10.4. *The constant C_* defined in (10.11) is strictly negative for $0 < r < r_c$.*

Proof. We can rewrite (10.12) in an equivalent form. Using (1.8), (10.12), $\alpha + \beta + \frac{2}{3} = 0$ and simple trigonometric formulas we obtain:

$$\frac{C_*}{9^{\frac{2}{3}}} = \frac{4\pi}{3} \sin \left(\pi \left(\alpha + \frac{2}{3} \right) \right) - 2 \cos \left(\pi \left(\alpha + \frac{1}{3} \right) \right) \log \left(\frac{r}{r_c} \right)$$

Using then (4.6) we can rewrite this expression as:

$$(10.13) \quad \frac{C_*}{9^{\frac{2}{3}}} = -\frac{4\pi}{3} \sin(\pi\zeta) - 2 \cos \left(\pi \left(\zeta + \frac{1}{3} \right) \right) \left[\frac{\log(2 \cos(\pi(\zeta + \frac{1}{3})))}{3\zeta} + \frac{\sqrt{3}\pi}{3} \right]$$

where $\zeta = -(\alpha + \frac{2}{3})$. Using Lemma 4.1 it follows that C_* would be negative for $0 < r < r_c$ if the right-hand side of (10.13) is negative for $\zeta \in (0, \frac{1}{6})$. This negativity can be proved as follows. The convexity of the function $\phi(x) = x \log(x) - x + 1$ for $x \geq 0$ implies the inequality $x \log(x) \geq x - 1$ for $x \in (0, 1)$, whence $\log(2A) \geq \frac{2A-1}{2A}$ for $A \in (0, \frac{1}{2})$. Since $\cos(\pi(\zeta + \frac{1}{3})) \in (0, \frac{1}{2})$ for $\zeta \in (0, \frac{1}{6})$ we then obtain $\log(2 \cos(\pi(\zeta + \frac{1}{3}))) \geq \frac{2 \cos(\pi(\zeta + \frac{1}{3})) - 1}{2 \cos(\pi(\zeta + \frac{1}{3}))}$.

Using this inequality in (10.13) we obtain $\frac{C_*}{9^{\frac{2}{3}}} \leq -\frac{4\pi}{3} \Phi(\zeta)$ with

$$\Phi(\zeta) = \sin(\pi\zeta) + \frac{2 \cos(\pi(\zeta + \frac{1}{3})) - 1}{4\pi\zeta} + \frac{\sqrt{3}}{2} \cos \left(\pi \left(\zeta + \frac{1}{3} \right) \right)$$

The concavity of the function $\cos(\pi(\zeta + \frac{1}{3}))$ for $\zeta \in (0, \frac{1}{6})$ implies $2 \cos(\pi(\zeta + \frac{1}{3})) - 1 \geq -2\pi\zeta \sin(\pi(\zeta + \frac{1}{3}))$. Then:

$$\Phi(\zeta) > \sin(\pi\zeta) - \frac{\sin(\pi(\zeta + \frac{1}{3}))}{2} + \frac{\sqrt{3}}{2} \cos \left(\pi \left(\zeta + \frac{1}{3} \right) \right) = 0 \quad \text{for } \zeta \in \left(0, \frac{1}{6} \right)$$

whence $C_* < 0$ for $0 < r < r_c$. \square

The proof of Proposition 10.3 is based on elementary arguments such as the representation of hypergeometric functions in terms of integral formulas and suitable contour deformations. However, the arguments are relatively cumbersome and the proof will be split in a sequence of Lemmas. We first derive a representation formula for C_* in terms of confluent hypergeometric functions.

Lemma 10.5. *Under the assumptions of Proposition 10.3 we have:*

$$(10.14) \quad \frac{C_*}{(9)^{\frac{2}{3}}} = - \lim_{R \rightarrow \infty} \int_0^R w \Delta_\alpha(w) dw - 2 \cos \left(\pi \left(\beta + \frac{1}{3} \right) \right) \log(r)$$

where:

$$(10.15) \quad \Delta_\alpha = U \left(-\alpha, \frac{2}{3}; -w^3 \right) U \left(-\beta, \frac{2}{3}; w^3 \right) - U \left(-\alpha, \frac{2}{3}; w^3 \right) U \left(-\beta, \frac{2}{3}; -w^3 \right)$$

Proof. The homogeneity of the integral in (10.11) implies that the integrals $\int_{\partial \mathcal{R}_\delta} [\cdot \cdot \cdot]$ are independent of δ . Moreover, integrating by parts and using (10.2), (10.3) we obtain that the integrals $\int_{\partial \mathcal{R}_{1,b}} [\cdot \cdot \cdot]$ take the same value for $0 < b < \infty$. Therefore, using the form of n :

$$(10.16) \quad \begin{aligned} C_* &= - \int_0^b [G_\alpha(x, r) D_v F_\beta(x, r) - F_\beta(x, r) D_v G_\alpha(x, r)] dx + \\ &\quad + \int_0^b [G_\alpha(x, -1) D_v F_\beta(x, -1) - F_\beta(x, -1) D_v G_\alpha(x, -1)] dx - \\ &\quad - \int_{-1}^r v G_\alpha(b, v) F_\beta(b, v) dv \end{aligned}$$

for any $b > 0$.

Using (4.8), (4.9) and (10.7) we obtain:

$$[G_\alpha(x, v) D_v F_\beta(x, v) - F_\beta(x, v) D_v G_\alpha(x, v)] = \frac{x^{\alpha+\beta-\frac{1}{3}}}{3} \left(\frac{v}{x^{\frac{1}{3}}} \right)^2 \Psi \left(\frac{v^3}{9x} \right)$$

where

$$(10.17) \quad \Psi(s) = \left[U \left(-\alpha, \frac{2}{3}; -s \right) DU \left(-\beta, \frac{2}{3}; s \right) + U \left(-\beta, \frac{2}{3}; s \right) DU \left(-\alpha, \frac{2}{3}; -s \right) \right]$$

Notice that the two first integrals on the right-hand side of (10.16) have the form:

$$(10.18) \quad \int_0^b \frac{x^{\alpha+\beta-\frac{1}{3}}}{3} \left(\frac{v}{x^{\frac{1}{3}}} \right)^2 \Psi \left(\frac{v^3}{9x} \right) dx$$

where v takes the values r and (-1) respectively. Using the change of variables $x = by$ as well as the fact that $\alpha + \beta + \frac{2}{3} = 0$ (cf. (10.6)) we can transform the integral (10.18) in:

$$(10.19) \quad \frac{9^{\frac{2}{3}}}{3} \int_0^1 \left(\frac{L^3}{9y} \right)^{\frac{2}{3}} \Psi \left(\frac{L^3}{9y} \right) \frac{dy}{y}$$

where $L = \frac{v}{b^{\frac{1}{3}}}$. Using (4.9), (4.10) we obtain $|\Psi(s)| \leq C |s|^{\alpha+\beta-1}$ whence $\left| s^{\frac{2}{3}} \Psi(s) \right| \leq C |s|^{\alpha+\beta+\frac{2}{3}-1} = \frac{C}{|s|}$. Therefore the integral in (10.19) can be estimated as $\frac{C}{L^3}$ whence the limit of the integrals in (10.18) converges to zero as $b \rightarrow 0$. It then follows from (10.16) that:

$$(10.20) \quad C_* = - \lim_{b \rightarrow 0} \int_{-1}^r v G_\alpha(b, v) F_\beta(b, v) dv$$

We now notice that (4.8), (4.9), (10.7) as well as $\alpha + \beta + \frac{2}{3} = 0$ yield:

$$v G_\alpha(b, v) F_\beta(b, v) = \frac{9^{\frac{2}{3}}}{(9b)^{\frac{1}{3}}} U \left(-\alpha, \frac{2}{3}; -\frac{v^3}{9b} \right) U \left(-\beta, \frac{2}{3}; \frac{v^3}{9b} \right) \frac{v}{(9b)^{\frac{1}{3}}}$$

Then, using the change of variables $w = \frac{v}{(9b)^{\frac{1}{3}}}$ and writing $R = \frac{1}{(9b)^{\frac{1}{3}}}$ we obtain:

$$C_* = -(9)^{\frac{2}{3}} \lim_{R \rightarrow \infty} \int_{-R}^{rR} U\left(-\alpha, \frac{2}{3}; -w^3\right) U\left(-\beta, \frac{2}{3}; w^3\right) w dw$$

Splitting the integral as $\int_{-R}^{rR} [\cdot \cdot \cdot] = \int_0^{rR} [\cdot \cdot \cdot] + \int_{-rR}^0 [\cdot \cdot \cdot] + \int_{-R}^{-rR} [\cdot \cdot \cdot]$ and using the change of variables $w \rightarrow (-w)$ in the last two integrals we obtain:

$$\begin{aligned} C_* &= -(9)^{\frac{2}{3}} \lim_{R \rightarrow \infty} \int_0^{rR} [U\left(-\alpha, \frac{2}{3}; -w^3\right) U\left(-\beta, \frac{2}{3}; w^3\right) \\ &\quad - U\left(-\alpha, \frac{2}{3}; w^3\right) U\left(-\beta, \frac{2}{3}; -w^3\right)] w dw \\ (10.21) \quad &- \lim_{R \rightarrow \infty} \int_{rR}^R U\left(-\alpha, \frac{2}{3}; w^3\right) U\left(-\beta, \frac{2}{3}; -w^3\right) w dw \end{aligned}$$

Using (4.9), (4.10) as well as $\alpha + \beta + \frac{2}{3} = 0$, we obtain:

$$U\left(-\alpha, \frac{2}{3}; w^3\right) U\left(-\beta, \frac{2}{3}; -w^3\right) w \sim 2 \cos\left(\pi\left(\beta + \frac{1}{3}\right)\right) (w)^{-1} \text{ as } w \rightarrow \infty$$

whence:

$$\lim_{R \rightarrow \infty} \int_{rR}^R U\left(-\alpha, \frac{2}{3}; w^3\right) U\left(-\beta, \frac{2}{3}; -w^3\right) w dw = -2 \cos\left(\pi\left(\beta + \frac{1}{3}\right)\right) \log(r)$$

Plugging this identity in (10.21) and replacing rR by R in the first limit on the right we obtain (10.14) and the result follows. \square

In order to compute the value of Δ_α in (10.15) we will use some representation formulas for the functions $U\left(-\alpha, \frac{2}{3}; -w^3\right)$, $U\left(-\beta, \frac{2}{3}; w^3\right)$:

Lemma 10.6. *Suppose that $0 < r < r_c$ and α, β are as in the Lemmas 4.1, 10.1 respectively. Then, the following representation formulas hold:*

$$(10.22) \quad U\left(-\alpha, \frac{2}{3}; w^3\right) = \frac{e^{w^3}}{\Gamma(-\alpha)} Q_2(w; \alpha + 1) \quad , \quad w > 0$$

$$(10.23) \quad U\left(-\beta, \frac{2}{3}; w^3\right) = \frac{w e^{w^3}}{\Gamma\left(\frac{1}{3} - \beta\right)} Q_1\left(w; \beta + \frac{2}{3}\right) \quad , \quad w > 0$$

where the functions Q_n are defined by means of:

$$(10.24) \quad Q_n(w; a) = \int_1^\infty e^{-w^3 t} \left(\frac{t}{t-1}\right)^a \frac{dt}{t^{\frac{2n}{3}}} \quad , \quad 0 < a < 1, \quad n > 0 \quad , \quad w > 0$$

Remark 10.7. *Notice that the functions Q_n are defined for noninteger values of n .*

Proof. We have:

$$(10.25) \quad U(a, b, z) = \frac{e^z}{\Gamma(a)} \int_1^\infty e^{-zt} (t-1)^{a-1} (t)^{b-a-1} dt \quad , \quad a > 0 \quad , \quad \operatorname{Re}(z) > 0$$

$$(10.26) \quad U(a, b, z) = z^{1-b} U(1+a-b, 2-b, z)$$

(cf. 13.2.6 and 13.1.29 from [1]). Using that $\alpha < 0$ we then obtain from (10.25):

$$(10.27) \quad U\left(-\alpha, \frac{2}{3}; w^3\right) = \frac{e^{w^3}}{\Gamma(-\alpha)} \int_1^\infty e^{-w^3 t} (t-1)^{-\alpha-1} (t)^{\alpha-\frac{1}{3}} dt \quad , \quad w > 0$$

whence (10.22) follows. On the other hand (10.26) yields

$$U\left(-\beta, \frac{2}{3}; w^3\right) = wU\left(\frac{1}{3} - \beta, \frac{4}{3}; w^3\right).$$

Using then that $\beta < \frac{1}{3}$ we obtain from (10.25):

$$(10.28) \quad U\left(-\beta, \frac{2}{3}; w^3\right) = \frac{we^{w^3}}{\Gamma\left(\frac{1}{3} - \beta\right)} \int_1^\infty e^{-w^3 t} (t-1)^{-(\beta+\frac{2}{3})} t^\beta dt, \quad w > 0$$

and (10.23) follows. \square

We now prove that the functions $Q_n(w; a)$ can be extended analytically for $w \neq 0$ and derive suitable representation formulas. To this end we need to give a precise definition of some analytic functions with branch points.

Definition 10.8. We define a branch of the function $\left(\frac{t}{t-1}\right)^a$ analytic in $\mathbb{C} \setminus [0, 1]$, which will be denoted as $\Phi(t; a)$, prescribing that $(s)^a = |s|^a e^{ia \arg(s)}$, with $\arg(s) \in (-\pi, \pi)$ for $s \in \mathbb{C} \setminus [0, \infty)$. On the other hand, we define the functions $t^{\frac{2n}{3}}$ with $n = 1, 2$ in a subset of a Riemann surface given by $\mathcal{S} = \{t = |t|e^{i\theta} : |t| \neq 0, \theta \in [-3\pi, 0]\}$. We set:

$$\left(|t|e^{i\theta}\right)^{\frac{2n}{3}} = |t|^{\frac{2n}{3}} e^{\frac{2n\theta i}{3}}$$

There exists a natural projection from the Riemann surface \mathcal{S} to \mathbb{C} . We will say that a contour Λ defined in \mathcal{S} surrounds the interval $[0, 1]$ if the projection of Λ into \mathbb{C} surrounds the interval $[0, 1]$.

Remark 10.9. We use the notation $t = |t|e^{i\theta}$ to denote points in the Riemann surface \mathcal{S} with a value of the phase θ . Notice that then the points $t_0 = 1 \in \mathcal{S}$ and $t_0 = e^{-2\pi i} \in \mathcal{S}$ are different points.

We can then obtain the following:

Lemma 10.10. The functions $Q_n(w; a)$ defined in (10.24) for $0 < a < 1$, $n = 1, 2$ and $w > 0$ can be extended analytically to the set $\{w \in \mathbb{C} : \text{Im}(w) > 0\}$ and continuously to the set $\{\text{Im}(w) \geq 0, w \neq 0\}$. Moreover, the following representation formulas hold for $w < 0$:

$$(10.29) \quad Q_n(w; a) = \int_\gamma e^{-w^3 t} \Phi(t; a) \frac{dt}{t^{\frac{2n}{3}}}, \quad n > 0, \quad 0 < a < 1$$

where the functions $\Phi(t)$ and $t^{\frac{2n}{3}}$ are as in Definition 10.8 and γ is a contour starting at $t = 1$ contained in the Riemann surface \mathcal{S} , surrounding the interval $[0, 1]$ and approaching asymptotically to $t = \infty \cdot e^{-3\pi i}$.

Remark 10.11. Notice that Lemma 10.10 yields a representation formula for

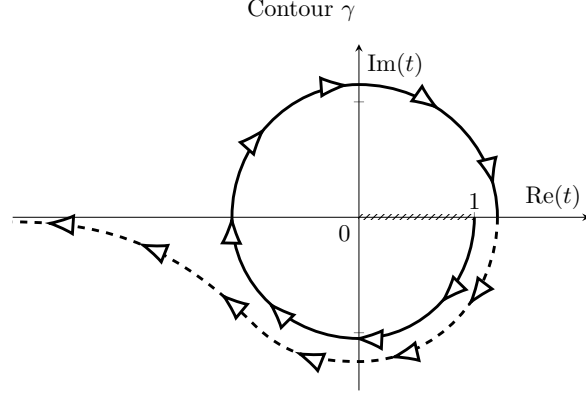
$$U\left(-\alpha, \frac{2}{3}; w^3\right), \quad U\left(-\beta, \frac{2}{3}; w^3\right) \quad \text{with } w < 0$$

by means of (10.22), (10.23).

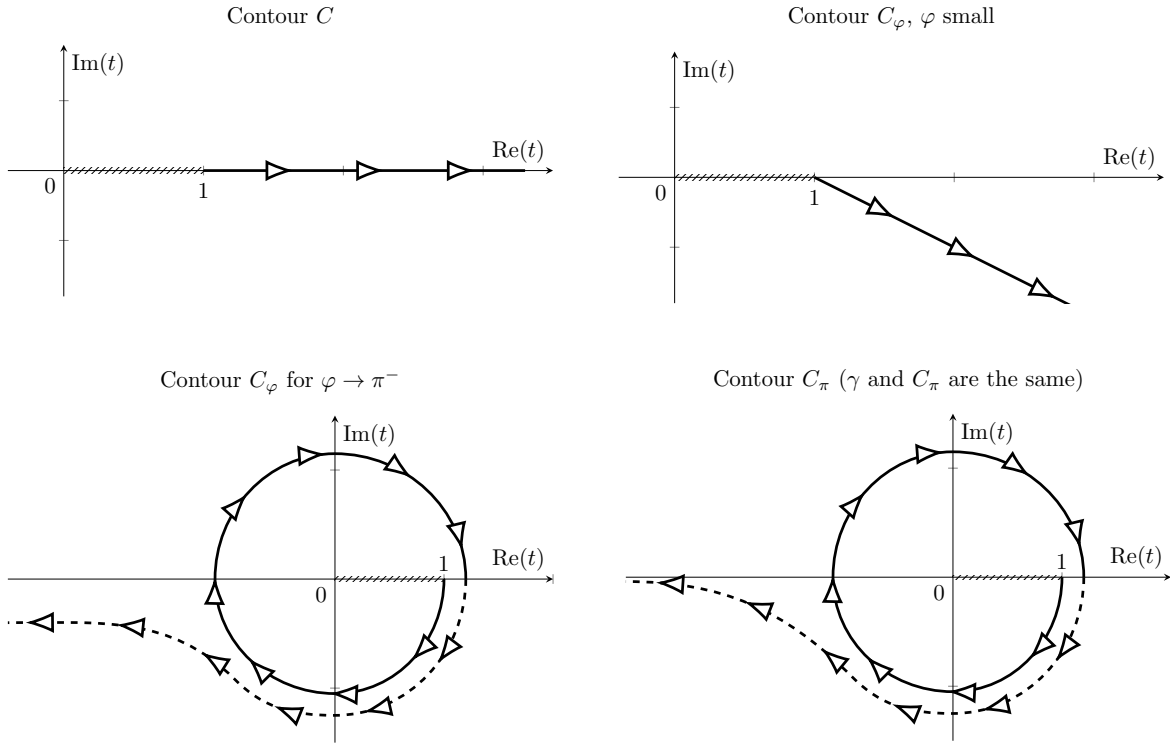
Proof. We can then rewrite (10.24) as:

$$(10.30) \quad Q_n(w; a) = \int_C e^{-w^3 t} \left(\frac{t}{t-1}\right)^a \frac{dt}{t^{\frac{2n}{3}}}, \quad w > 0$$

where $C \subset \mathcal{S}$ is the line connecting 1 and ∞ given by $\{t : t = 1 + \rho e^{i\theta}, \rho \in [0, \infty), \theta = 0\}$. In order to extend analytically the function $Q_n(w; a)$ to the region $\{\text{Im}(w) > 0\}$ we use

FIGURE 1. Contour γ

(10.30) with $w = |w|e^{\varphi i}$ with $\varphi \in [0, \pi]$. In order to ensure the convergence of the integrals we can modify the contour of integration to a new contour C_φ which connects $t = 1$ with $t = \infty$ and for sufficiently large $|t|$, say $|t| \geq 3$ is just the line $\{t : t = |t|e^{-3\varphi i}\}$. Therefore, if φ varies from 0 to π we would obtain that this line which describes the asymptotics of the new contours C_φ is just the line $\{t : t = |t|e^{-3\pi i}\}$ of the set \mathcal{S} . Notice that this contour deformation must be made avoiding intersections of the new contours C_φ with the interval $[0, 1]$. The function $\Phi(t; a)$ can be defined in \mathcal{S} in a natural manner using just its definition in $\mathbb{C} \setminus [0, 1]$. After concluding the deformation of the contours we obtain a representation formula for $Q_n(w; a)$ with $w < 0$ having the form (10.29). \square

FIGURE 2. Contours C_φ for some values of φ

We can now find a representation formula for the limit $\lim_{R \rightarrow \infty} \int_0^R w \Delta_\alpha(w) dw$ in (10.14). We first define the following family of auxiliary functions:

Definition 10.12. Suppose that $0 < A < 1$, $0 < B < 1$, $n > 0$, $m > 0$. We define:

$$\Psi(t, x; A, B, n, m) = \frac{\Phi(t; A) \Phi(x; B)}{t^{\frac{2m}{3}} x^{\frac{2n}{3}}} + \frac{\Phi(t; B) \Phi(x; A)}{x^{\frac{2m}{3}} t^{\frac{2n}{3}}}, \quad t \in \mathcal{S}, \quad x \in \mathcal{S}$$

where the functions Φ are as in Definition 10.8 and the power laws $s \rightarrow (s)^a$ are computed in the portion of Riemann surface \mathcal{S} as in Definition 10.8.

We then have:

Lemma 10.13. Suppose that we define the following real functions for $x > 1$, $0 < t < 1$:

$$(10.31) \quad \Psi_1(t, x; A, B, n, m) = \left(\frac{t}{1-t} \right)^A \left(\frac{x}{x-1} \right)^B \frac{1}{t^{\frac{2m}{3}} x^{\frac{2n}{3}}}$$

$$(10.32) \quad \Psi_2(t, x; A, B, n, m) = \left(\frac{t}{1-t} \right)^B \left(\frac{x}{x-1} \right)^A \frac{1}{t^{\frac{2n}{3}} x^{\frac{2m}{3}}}$$

We define a contour Λ in \mathcal{S} as $\Lambda = \Lambda_1 \cup \Lambda_2$ where:

$$(10.33) \quad \Lambda_1 = \left\{ t \in \mathcal{S} : t = \frac{e^{i\theta}}{2}, \quad \theta \in [0, -2\pi] \right\}$$

and Λ_2 is a contour connecting $\frac{e^{-2\pi i}}{2}$ with $t = \infty \cdot e^{-3\pi i}$. We define:

$$(10.34) \quad K_m(A) = \left(-e^{i\pi A} + e^{-i\pi A} e^{\frac{4\pi m i}{3}} - e^{i\pi A} e^{\frac{4\pi m i}{3}} \right), \quad 0 < A < 1, \quad m > 0$$

Let $A = \alpha + 1$, $B = \beta + \frac{2}{3}$, $n = 1$, $m = 2$. Let

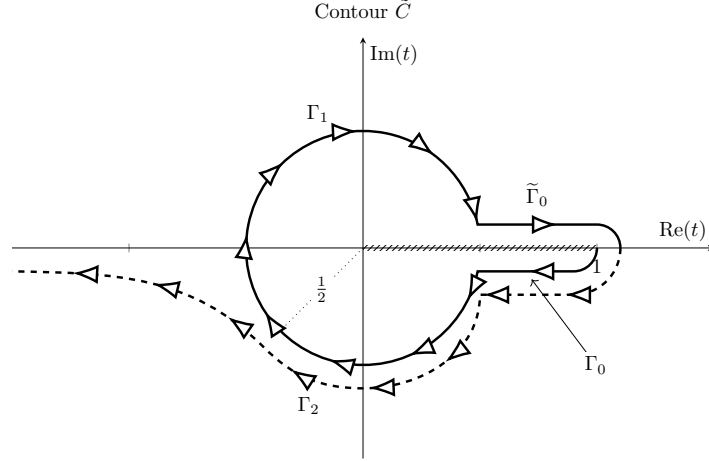
$$(10.35) \quad G(t, x; A, B, n, m) = K_m(A) \Psi_1 + K_n(B) \Psi_2$$

Then the function $\frac{|G(t, x)|}{|t-x|}$ is integrable in $(t, x) \in [\frac{1}{2}, 1] \times (1, \infty)$ and we have:

$$(10.36) \quad \begin{aligned} & \lim_{R \rightarrow \infty} \int_0^R w \Delta_\alpha(w) dw \\ &= -\frac{\sin(\pi\alpha)}{3\pi} \left[\int_1^\infty dx \int_\Lambda dt \frac{\Psi(t, x; A, B, n, m)}{(x-t)} + \right. \\ & \quad \left. + \int_1^\infty dx \int_{\frac{1}{2}}^1 dt \frac{G(t, x; A, B, n, m)}{(x-t)} \right] \end{aligned}$$

Remark 10.14. The rationale behind the definition of the contour Λ is to avoid the contour of integration approaching the origin $t = 0$, because the function Ψ is not integrable there for the range of parameters required.

Remark 10.15. We introduce here a contour \tilde{C} for further reference. This contour consists in the limit of contours approaching the segment $[\frac{1}{2}, 1]$, connecting the points $t = 1$ with $t = \frac{1}{2}$ with $t = |t| e^{i0}$ and $\text{Im}(t) \rightarrow 0^-$. We continue the contour by means of the contour Λ_1 defined in (10.33). It is then followed by a contour obtained as limit of contours converging to the interval $[\frac{1}{2}, 1]$ with $\text{Im}(t) > 0$, $t = |t| e^{-2\pi i}$ and connecting the points $t = \frac{1}{2}, 1$. We then continue the contour by a segment converging to $[\frac{1}{2}, 1]$ with $\text{Im}(t) < 0$, $t = |t| e^{-2\pi i}$, connecting $t = 1, \frac{1}{2}$. The last part of the contour is then the contour Λ_2 in the statement of the Lemma.

FIGURE 3. Contour \tilde{C}

Proof. Using (10.15), (10.22), (10.23), (10.29) we obtain the representation formula:

$$\begin{aligned}
 (10.37) \quad \Delta_\alpha &= \frac{w}{\Gamma(-\alpha)\Gamma(\frac{1}{3}-\beta)} \left[\int_1^\infty e^{-w^3x} \left(\frac{x}{x-1} \right)^{\beta+\frac{2}{3}} \frac{dx}{x^{\frac{2}{3}}} \int_\gamma e^{w^3t} \Phi(t; \alpha+1) \frac{dt}{t^{\frac{4}{3}}} \right. \\
 &\quad \left. + \int_1^\infty e^{-w^3x} \left(\frac{x}{x-1} \right)^{\alpha+1} \frac{dx}{x^{\frac{4}{3}}} \int_\gamma e^{w^3t} \Phi\left(t; \beta + \frac{2}{3}\right) \frac{dt}{t^{\frac{2}{3}}} \right] \\
 &\equiv \frac{w}{\Gamma(-\alpha)\Gamma(\frac{1}{3}-\beta)} (J_1 + J_2)
 \end{aligned}$$

where $\gamma \in \mathcal{S}$ is as in Lemma 10.10. We can deform the contour of integration γ to the contour \tilde{C} introduced in Remark 10.15. Using the Definition of the functions $\Phi(t; \alpha+1)$, $t^{\frac{4}{3}}$ we obtain the following formula for the integral $\int_\gamma = \int_{\tilde{C}}$ in the definition of J_1 :

$$\begin{aligned}
 &\int_\gamma e^{w^3t} \Phi(t; \alpha+1) \frac{dt}{t^{\frac{4}{3}}} \\
 &= \int_\Lambda e^{w^3t} \Phi(t; \alpha+1) \frac{dt}{t^{\frac{4}{3}}} \\
 &\quad + \left(-e^{i\pi(\alpha+1)} + e^{-i\pi(\alpha+1)} e^{\frac{8\pi i}{3}} - e^{i\pi(\alpha+1)} e^{\frac{8\pi i}{3}} \right) \int_{\frac{1}{2}}^1 e^{w^3t} \left(\frac{t}{1-t} \right)^{\alpha+1} \frac{dt}{t^{\frac{4}{3}}} \\
 &= \int_\Lambda e^{w^3t} \Phi(t; \alpha+1) \frac{dt}{t^{\frac{4}{3}}} + K_2(\alpha+1) \int_{\frac{1}{2}}^1 e^{w^3t} \left(\frac{t}{1-t} \right)^{\alpha+1} \frac{dt}{t^{\frac{4}{3}}}
 \end{aligned}$$

where we use that:

$$(10.38) \quad \lim_{\varepsilon \rightarrow 0^+} \Phi(x + \varepsilon i) = \left(\frac{x}{1-x} \right)^a e^{-i\pi a}, \quad x \in (0, 1)$$

$$(10.39) \quad \lim_{\varepsilon \rightarrow 0^+} \Phi(x - \varepsilon i) = \left(\frac{x}{1-x} \right)^a e^{i\pi a}, \quad x \in (0, 1)$$

A similar argument gives:

$$\begin{aligned} & \int_{\gamma} e^{w^3 t} \Phi \left(t; \beta + \frac{2}{3} \right) \frac{dt}{t^{\frac{2}{3}}} \\ &= \int_{\Lambda} e^{w^3 t} \Phi \left(t; \beta + \frac{2}{3} \right) \frac{dt}{t^{\frac{2}{3}}} + K_1 \left(\beta + \frac{2}{3} \right) \int_{\frac{1}{2}}^1 e^{-w^3 t} \left(\frac{t}{1-t} \right)^{\beta + \frac{2}{3}} \frac{dt}{t^{\frac{2}{3}}} \end{aligned}$$

We then obtain:

$$(10.40) \quad \begin{aligned} J_1 + J_2 &= \int_1^{\infty} dx \int_{\Lambda} dt e^{-w^3(x-t)} \Psi(t, x; A, B, n, m) + \\ &+ \int_1^{\infty} dx \int_{\frac{1}{2}}^1 dt e^{-w^3(x-t)} G(t, x; A, B, n, m) \end{aligned}$$

Notice that the integrability of $\frac{|G(t, x)|}{|t-x|}$ near $x = 1, t = 1$ is a consequence of the fact that we have $K_2(\alpha + 1) + K_1(\beta + \frac{2}{3}) = 0$ and this implies the inequality:

$$(10.41) \quad \begin{aligned} & \left| G \left(t, x; \alpha + 1, \beta + \frac{2}{3}, 1, 2 \right) \right| \\ & \leq C \left(\frac{1}{(1-t)^{\alpha+1} (x-1)^{\beta+\frac{2}{3}}} + \frac{1}{(x-1)^{\alpha+1} (1-t)^{\beta+\frac{2}{3}}} \right) |x-t| \end{aligned}$$

and the desired integrability follows since $(\alpha + 1) < 1, (\beta + \frac{2}{3}) < 1$. The fact that $K_2(\alpha + 1) + K_1(\beta + \frac{2}{3}) = 0$ follows from the following computation which is a consequence of the fact that $\beta + \frac{2}{3} = -\alpha$:

$$K_2(\alpha + 1) + K_1 \left(\beta + \frac{2}{3} \right) = (e^{i\pi\alpha} - e^{-i\pi\alpha}) \left(1 + e^{\frac{4\pi i}{3}} + e^{\frac{8\pi i}{3}} \right) = 0$$

Integrating (10.37) in $[0, R]$ after multiplying by w , using (10.40) and taking the limit $R \rightarrow \infty$ we obtain (10.36), using also the formula $\Gamma(\beta + \frac{2}{3}) \Gamma(\frac{1}{3} - \beta) = \frac{\pi}{\sin(\pi(\beta + \frac{2}{3}))} = -\frac{\pi}{\sin(\pi\alpha)}$ (cf. 6.1.17 in [1]) we obtain (10.36). \square

We now study the properties of the functions defined by means of the right-hand side of (10.36). More precisely, we define the following subset of \mathbb{R}^4 :

$$(10.42) \quad \mathcal{K} = \left\{ (A, B, n, m) : 0 < A, B < 1, n > 0, m > 0, \frac{2(n+m)}{3} > 1, K_m(A) + K_n(B) = 0 \right\}$$

where the functions $K_m(A), K_n(B)$ are defined as in (10.34). Therefore:

Lemma 10.16. *The formula:*

$$(10.43) \quad \begin{aligned} Q(A, B, n, m) &= \left[\int_1^{\infty} dx \int_{\Lambda} dt \frac{\Psi(t, x; A, B, n, m)}{(x-t)} + \right. \\ &\quad \left. + \int_1^{\infty} dx \int_{\frac{1}{2}}^1 dt \frac{G(t, x; A, B, n, m)}{(x-t)} \right] \end{aligned}$$

defines a continuous function in the set \mathcal{K} in (10.42) where the contour Λ is as in Lemma 10.13, Ψ is as in Definition 10.12, and $G(t, x; A, B, n, m)$ as in (10.35) with $K_m(A), K_n(B)$ as in (10.34) and Ψ_1, Ψ_2 are as in (10.31), (10.32).

Suppose that in addition to $(A, B, n, m) \in \mathcal{K}$ we have $A + B < 1$. Then the following representation formula holds:

$$(10.44) \quad Q(A, B, n, m) = \int_1^\infty dx \int_{\tilde{C}} dt \frac{\Psi(t, x; A, B, n, m)}{(x - t)}$$

where \tilde{C} is the contour introduced in Remark 10.15.

Proof. Since $(A, B, n, m) \in \mathcal{K}$ it follows from the definition of G , $K_m(A)$, $K_n(B)$, and Ψ_1, Ψ_2 that the inequality (10.41) holds. Therefore the formula (10.43) defines a continuous function Q in \mathcal{K} . Suppose that $A + B < 1$. Then, since the contour \tilde{C} is contained in $\operatorname{Re}(t) \leq 1$ we obtain, using Definition 10.12 the following estimate for $t \in \Lambda \cap \{|t| \leq 2\}$, $x \in \{|x| \leq 2\}$:

$$\left| \frac{\Psi(t, x; A, B, n, m)}{x - t} \right| \leq K \left(\frac{1}{(1 - t)^{A+A^*} (x - 1)^{B+B^*}} + \frac{1}{(1 - t)^{B+B^*} (x - 1)^{A+A^*}} \right)$$

where $A^* + B^* = 1$, $A + A^* < 1$, $B + B^* < 1$ and K is independent of A, B, n, m . Notice that the choice of A^*, B^* is possible because $A + B < 1$. Then, the function $\left| \frac{\Psi(t, x; A, B, n, m)}{x - t} \right|$ is integrable in a neighbourhood of $t = x = 1$, and using the definition of the functions $\Phi, (t)^a$ in Definition 10.8 we obtain exactly the numerical factors in the functions $K_m(A)$, $K_n(B)$ in the integral in the contour \tilde{C} . This gives the representation formula (10.44). \square

We now describe the structure of the set \mathcal{K} in the neighbourhood of a point $(A_0, B_0, 1, 2)$ with $A_0 + B_0 = 1$, $A_0 > 0$, $B_0 > 0$. Notice that a computation similar to (10.44) shows that such points are contained in \mathcal{K} . We have:

Lemma 10.17. *For each positive A_0, B_0 satisfying $A_0 + B_0 = 1$ there exists $\delta > 0$ and two differentiable functions $\Theta_1(A, B)$, $\Theta_2(A, B)$ defined in $|A - A_0| + |B - B_0| < \delta$ such that $\Theta_1(A_0, B_0) = 1$, $\Theta_2(A_0, B_0) = 2$ and such that $(A, B, \Theta_1(A, B), \Theta_2(A, B))$ belong to \mathcal{K} . Moreover, we have:*

$$(10.45) \quad \lim_{\varepsilon \rightarrow 0} \frac{\Theta_1(A_0, B_0 - \varepsilon) - 1}{\varepsilon} = -\frac{3}{8} \left(1 + \sqrt{3} \cot(\pi A_0) \right)$$

$$(10.46) \quad \lim_{\varepsilon \rightarrow 0} \frac{\Theta_2(A_0, B_0 - \varepsilon) - 2}{\varepsilon} = -\frac{3}{8} \left(1 + \sqrt{3} \cot(\pi A_0) \right)$$

Proof. The set \mathcal{K} is defined by means of the equation $K_m(A) + K_n(B) = 0$. Taking the real and imaginary part of this equation we obtain that it is equivalent to the two real equations $F_1 = F_2 = 0$ with

$$\begin{aligned} & F_1(A, B, n, m) \\ &= -\cos(\pi A) + \cos\left(\frac{4\pi m}{3} - \pi A\right) - \cos\left(\frac{4\pi m}{3} + \pi A\right) \\ & \quad - \cos(\pi B) + \cos\left(\frac{4\pi n}{3} - \pi B\right) - \cos\left(\frac{4\pi n}{3} + \pi B\right) \\ & F_2(A, B, n, m) \\ &= -\sin(\pi A) + \sin\left(\frac{4\pi m}{3} - \pi A\right) - \sin\left(\frac{4\pi m}{3} + \pi A\right) \\ & \quad - \sin(\pi B) + \sin\left(\frac{4\pi n}{3} - \pi B\right) - \sin\left(\frac{4\pi n}{3} + \pi B\right) \end{aligned}$$

The existence of the functions Θ_1, Θ_2 is just a consequence of the Implicit Function Theorem. Indeed, since we have $F_1(A_0, B_0, 1, 2) = F_2(A_0, B_0, 1, 2) = 0$ we only need to check

that $\frac{\partial(F_1, F_2)}{\partial(n, m)}(A_0, B_0, 1, 2) \neq 0$. This is equivalent to proving that:

$$\det \begin{pmatrix} -\sin\left(\frac{4\pi}{3} - \pi B_0\right) + \sin\left(\frac{4\pi}{3} + \pi B_0\right) & \cos\left(\frac{4\pi}{3} - \pi B_0\right) - \cos\left(\frac{4\pi}{3} + \pi B_0\right) \\ -\sin\left(\frac{8\pi}{3} - \pi A_0\right) + \sin\left(\frac{8\pi}{3} + \pi A_0\right) & \cos\left(\frac{8\pi}{3} - \pi A_0\right) - \cos\left(\frac{8\pi}{3} + \pi A_0\right) \end{pmatrix} \neq 0$$

Using that $B_0 = (1 - A_0)$ and elementary trigonometric formulas we obtain that this condition equivalent to:

$$0 \neq \det \begin{pmatrix} 2 \sin(\pi(1 - A_0)) \cos\left(\frac{4\pi}{3}\right) & 2 \sin\left(\frac{4\pi}{3}\right) \sin(\pi(1 - A_0)) \\ 2 \cos\left(\frac{8\pi}{3}\right) \sin(\pi A_0) & 2 \sin\left(\frac{8\pi}{3}\right) \sin(\pi A_0) \end{pmatrix} = -2\sqrt{3} \sin^2(\pi A_0)$$

which holds for $A_0 \in (0, 1)$.

In order to obtain the asymptotics (10.45), (10.46) we argue as follows. We write $n = 1 + \delta_1$, $m = 2 + \delta_2$. Using Taylor's to approximate the equation $K_m(A) + K_n(B) = 0$ for $A = A_0$, $B = B_0 - \varepsilon$, we obtain the following approximation to the linear order:

$$\begin{aligned} 0 &= -\frac{8}{3} \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) \sin(\pi A_0) \delta_2 + \left(\sin(\pi A_0) + \sqrt{3} \cos(\pi A_0) \right) \varepsilon \\ &\quad - \frac{8}{3} \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i \right) \sin(\pi A_0) \delta_1 \end{aligned}$$

The imaginary part of this equation implies, to the linear order, that $\delta_1 = \delta_2$. We then obtain the approximation $\delta_1 = \delta_2 = -\frac{3}{8} (1 + \sqrt{3} \cot(\pi A_0)) \varepsilon$ as $\varepsilon \rightarrow 0$. This gives (10.45), (10.46). This computation can be made fully rigorous by means of a standard application of the Implicit Function Theorem. We will skip the details. \square

We can obtain now a representation formula for $Q(A, B, n, m)$ with $(A, B, n, m) \in \mathcal{K}$, $A + B < 1$.

Lemma 10.18. *Let $(A, B, n, m) \in \mathcal{K}$ with $A + B < 1$. Suppose that $Q(A, B, n, m)$ is as in Lemma 10.16. Then the following representation formula holds:*

$$(10.47) \quad Q(A, B, n, m) = -2\pi i \int_1^\infty \left(1 + e^{\frac{4\pi n i}{3}} + e^{\frac{4\pi m i}{3}} \right) \left(\frac{t}{t-1} \right)^{A+B} \frac{dt}{t^{\frac{2(n+m)}{3}}}$$

Remark 10.19. *Notice that $\frac{2(n+m)}{3} > 1$ if $(A, B, n, m) \in \mathcal{K}$. Therefore the integral on the right of (10.47) is well defined.*

Proof. Under the assumptions of the Lemma, (10.44) holds. We can now deform the contour of integration $\tilde{C} \in \mathcal{S}$ to a new contour given by the line $L = \{t \in \mathcal{S} : 1 + re^{-i\delta} : r \geq 0\}$ where $\delta > 0$ is a small number. The deformation is made by means of a family of contours which behave asymptotically for large $|t|$ as the line $\{t = re^{i\theta}\}$ with θ varying from -3π to $-\delta$. In the process of deformation the deforming contours must cross any point $t = x$ with $x \in (1, \infty)$. This gives a contribution due to Residue Theorem equal to $-2\pi i \Psi(e^{-2\pi i} x, x)$, where the notation $\Psi(e^{-2\pi i} x, x)$ indicates that the function Ψ must be evaluated at that particular point of the Riemann surface \mathcal{S} . Then:

$$(10.48) \quad Q(A, B, n, m) = -2\pi i \int_1^\infty \Psi(e^{-2\pi i} x, x; A, B, n, m) dx - \int_1^\infty dx \int_L dt \frac{\Psi(t, x; A, B, n, m)}{(t-x)}$$

In order to compute the last integral in (10.48) we argue as follows. Exchanging the role of x and t we obtain:

$$\int_1^\infty dx \int_L dt \frac{\Psi(t, x; A, B, n, m)}{(t-x)} = \int_1^\infty dt \int_L dx \frac{\Psi(x, t; A, B, n, m)}{(x-t)}$$

Using now that $\Psi(x, t; A, B, n, m) = \Psi(t, x; A, B, n, m)$ we obtain:

$$(10.49) \quad \int_1^\infty dx \int_L dt \frac{\Psi(t, x; A, B, n, m)}{(t-x)} = \int_1^\infty dt \int_L dx \frac{\Psi(t, x; A, B, n, m)}{(x-t)}$$

In order to transform the last integral in the original one we deform the contour L to $(1, \infty)$ and the contour $(1, \infty)$ to L . In the process of deformation one the contour L must cross the point $x = t$ yielding a contribution due to Residue Theorem. Then:

$$(10.50) \quad \begin{aligned} & \int_1^\infty dt \int_L dx \frac{\Psi(t, x; A, B, n, m)}{(x-t)} \\ &= \int_L dt \int_1^\infty dx \frac{\Psi(t, x; A, B, n, m)}{(x-t)} + 2\pi i \int_1^\infty \Psi(t, t; A, B, n, m) dt \end{aligned}$$

Combining (10.49), (10.50) we obtain:

$$\int_1^\infty dx \int_L dt \frac{\Psi(t, x; A, B, n, m)}{(t-x)} = \pi i \int_1^\infty \Psi(t, t; A, B, n, m) dt$$

Plugging this formula into (10.48) and relabelling the name of the variable in the first integral of the right of (10.48) we obtain:

$$Q(A, B, n, m) = -2\pi i \int_1^\infty \Psi(e^{-2\pi i} t, t; A, B, n, m) dt - \pi i \int_1^\infty \Psi(t, t; A, B, n, m) dt$$

We now use that, taking into account the analyticity of the function $\Phi(t; a) = \left(\frac{t}{t-1}\right)^a$ in $C \setminus [0, 1]$ we have:

$$\Psi(t, t; A, B, n, m) = 2 \left(\frac{t}{t-1}\right)^{A+B} \frac{1}{t^{\frac{2(n+m)}{3}}}, \quad \arg(t) \in [-\delta, 0]$$

$$\Psi(e^{-2\pi i} t, t; A, B, n, m) = \left[e^{\frac{4\pi n i}{3}} + e^{\frac{4\pi m i}{3}} \right] \left(\frac{t}{t-1}\right)^{A+B} \frac{1}{t^{\frac{2(n+m)}{3}}}, \quad \arg(t) \in [-\delta, 0]$$

if $\delta > 0$ is sufficiently small, whence (10.47) follows. \square

We now take the limit of $Q(A, B, n, m)$ as $(A+B) \rightarrow 1^-$.

Lemma 10.20. *Suppose that $0 < A_0 < 1$, $0 < B_0 < 1$ satisfy $A_0 + B_0 = 1$. Then:*

$$Q(A_0, B_0, 1, 2) = \pi^2 \left(1 + \sqrt{3} \cot(\pi A_0)\right)$$

Proof. Lemmas 10.16, 10.17 imply:

$$(10.51) \quad Q(A_0, B_0, 1, 2) = \lim_{\varepsilon \rightarrow 0^+} Q(A_0, B_0 - \varepsilon, \Theta_1(A_0, B_0 - \varepsilon), \Theta_2(A_0, B_0 - \varepsilon))$$

Using (10.45), (10.46) as well as the fact that $1 + e^{\frac{4\pi i}{3}} + e^{\frac{8\pi i}{3}} = 0$ we obtain:

$$(10.52) \quad \begin{aligned} \frac{1}{\varepsilon} \left(1 + e^{\frac{4\pi \Theta_1(A_0, B_0 - \varepsilon) i}{3}} + e^{\frac{4\pi \Theta_2(A_0, B_0 - \varepsilon) i}{3}} \right) &\rightarrow -\frac{3 \cdot 4\pi i \left(e^{\frac{4\pi i}{3}} + e^{\frac{8\pi i}{3}} \right)}{3 \cdot 8} \left(1 + \sqrt{3} \cot(\pi A_0) \right) \\ &= \frac{\pi i}{2} \left(1 + \sqrt{3} \cot(\pi A_0) \right) \end{aligned}$$

as $\varepsilon \rightarrow 0$. Notice that we use $e^{\frac{4\pi i}{3}} + e^{\frac{8\pi i}{3}} = -1$. We can now take the limit in (10.51) using (10.47). Notice that the main contribution to the integral is due to the region where t is close to one. We can then split the integral as $\int_1^{1+\delta} + \int_{1+\delta}^\infty$ with $\delta > 0$ small. The second term is

bounded as $C_\delta \varepsilon$ for each $\delta > 0$ due to (10.52) and in the first we can approximate t by 1 in $t^{\frac{2(n+m)}{3}}$ and t^{A+B} . Then, using again (10.52) we obtain:

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0^+} Q(A_0, B_0 - \varepsilon, \Theta_1(A_0, B_0 - \varepsilon), \Theta_2(A_0, B_0 - \varepsilon)) \\
&= -2\pi i \lim_{\varepsilon \rightarrow 0^+} \int_1^{1+\delta} \left(1 + e^{\frac{4\pi\Theta_1 i}{3}} + e^{\frac{4\pi\Theta_2 i}{3}}\right) \left(\frac{t}{t-1}\right)^{A_0+B_0-\varepsilon} \frac{dt}{t^{\frac{2(n+m)}{3}}} \\
&= \pi^2 \left(1 + \sqrt{3} \cot(\pi A_0)\right) \lim_{\varepsilon \rightarrow 0^+} \varepsilon \int_1^{1+\delta} \left(\frac{1}{t-1}\right)^{1-\varepsilon} \\
&= \pi^2 \left(1 + \sqrt{3} \cot(\pi A_0)\right)
\end{aligned}$$

and the result follows. \square

We can now compute C_* and finish the Proof of Proposition 10.3

End of the Proof of Proposition 10.3. Using (10.36), (10.43) we obtain

$$\lim_{R \rightarrow \infty} \int_0^R w \Delta_\alpha(w) dw = -\frac{\sin(\pi\alpha)}{3\pi} Q\left(1 + \alpha, \beta + \frac{2}{3}, 1, 2\right).$$

Lemma 10.20 yields

$$\begin{aligned}
\lim_{R \rightarrow \infty} \int_0^R w \Delta_\alpha(w) dw &= -\frac{\pi \sin(\pi\alpha)}{3} \left(1 + \sqrt{3} \cot(\pi(1 + \alpha))\right) \\
&= -\frac{\pi}{3} \left(\sin(\pi\alpha) + \sqrt{3} \cos(\pi\alpha)\right),
\end{aligned}$$

whence:

$$\frac{C_*}{(9)^{\frac{2}{3}}} = \frac{\pi}{3} \left(\sin(\pi\alpha) + \sqrt{3} \cos(\pi\alpha)\right) - 2 \cos\left(\pi\left(\beta + \frac{1}{3}\right)\right) \log(r)$$

\square

11. DEFINITION OF SOME DIFFERENTIAL OPERATORS.

Our goal now is to obtain suitable adjoint operators in the sense of Definition 8.1 for (1.3), (1.4), with the boundary conditions (5.8), (5.9), (5.11), (6.1). These operators will act over the class of continuous functions on a topological space X . The action of the operators will be given by a differential operator \mathcal{L} with suitable boundary conditions. In this Section we give the precise definitions of X and \mathcal{L} .

Definition 11.1. We define as X_0 the set obtained identifying the subset of points $[0, \infty) \times (-\infty, \infty)$ such that $(x, v) = (0, -v)$ and $(x, v) = (0, rv)$, $v > 0$. We then define $X = X_0 \cup \{\infty\}$, and we endowed it with the natural topology inherited from \mathbb{R}^2 complemented with the following set of neighbourhoods of the point ∞ :

$$\mathcal{O}_M = \{(x, v) \in [0, \infty) \times (-\infty, \infty) : v < -M \text{ or } v > rM \text{ or } x > M\} \quad , \quad M > 0$$

The set X is a topological compact set. The continuous functions of this space can be identified with the bounded continuous functions φ in $[0, \infty) \times (-\infty, \infty)$ such that

$$(11.1) \quad \varphi(0, -v) = \varphi(0, rv), \quad v > 0$$

and such that the limit $\lim_{x+|v| \rightarrow \infty} \varphi(x, v)$ exists. We will denote this set of functions as $C(X)$. Notice that a function $\varphi \in C(X)$ defines a function in $C(\mathcal{U})$ satisfying (11.1). We will use the same notation φ to refer to both functions for the sake of simplicity.

We need to introduce some local directionality in a neighbourhood of each point of $X \setminus \{(0, 0), \infty\}$ in order to compute directional limits.

Definition 11.2. Given two points $(x_1, v_1), (x_2, v_2) \in X \setminus [\{(x, 0) : x \geq 0\} \cup \{\infty\}]$. We will say that (x_1, v_1) is to the left of (x_2, v_2) and we will $(x_1, v_1) \ll (x_2, v_2)$ if $x_1 \operatorname{sgn}(v_1) < x_2 \operatorname{sgn}(v_2)$.

Notice that the previous definition just means that $(x_1, v_1) \ll (x_2, v_2)$ in one of the following three cases: (i) If $v_1 > 0$ and $v_2 > 0$ we have $x_1 < x_2$. (ii) $v_1 < 0 < v_2$. (iii) If $v_1 < 0$ and $v_2 < 0$ we have $x_1 > x_2$.

Definition 11.3. Given a point $(x_0, v_0) \in X \setminus \{(x, 0) : x \geq 0\} \cup \{\infty\}$ and a neighbourhood \mathcal{B} of (x_0, v_0) in the topological space X we define the left neighbourhood $\mathcal{B}^-(x_0, v_0)$ as:

$$\mathcal{B}^-(x_0, v_0) = \{(x, v) \in \mathcal{B} : (x, v) \ll (x_0, v_0)\}$$

Remark 11.4. Notice that the neighbourhood \mathcal{B} must be understood as a neighbourhood of the topological space X . In particular, if $(x_0, v_0) = (0, v_0)$ any neighbourhood of (x_0, v_0) contains points (x, v) with $v > 0$ and $v < 0$.

Definition 11.5. We will say that $L \subset X$ is a vertical segment if it has the form $\{(x_0, v) \in X : v \in (\alpha, \beta)\}$ for some $x_0 \geq 0$, $\alpha, \beta \in \mathbb{R}$ with $\alpha \cdot \beta > 0$, $\alpha < \beta$. Given two vertical segments L_1, L_2 we will say that L_1 is to the left of L_2 if for any $(x_k, v_k) \in L_k$ with $k = 1, 2$ we have $(x_1, v_1) \ll (x_2, v_2)$. We will then write $L_1 \ll L_2$. We will say that $L \subset X$ is a horizontal segment if it has the form $\{(x, v_0) \in X : x \in (\alpha, \beta)\}$ for some $v_0 \in \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$ with $0 \leq \alpha < \beta$.

It will be convenient to define a suitable concept of convergence in the set of segments.

Definition 11.6. Given two segments L_1, L_2 in X we define a distance between them as:

$$(11.2) \quad \operatorname{dist}_H(L_1, L_2) = \inf \{\operatorname{dist}((x, v), L_2) : (x, v) \in L_1\}$$

The action of the operators Ω_σ on smooth functions supported in \mathcal{U} (cf. (5.2)) is given by the differential operator

$$(11.3) \quad \mathcal{L} = D_v^2 + v D_x$$

where the operator \mathcal{L} will be defined in the sense of distributions as indicated later. Nevertheless, the operators Ω_σ will differ in the different cases (cf. (5.8), (5.9), (5.10), (6.1)) in its domain of definition which will encode the asymptotic behaviour of φ near the singular point $(x, v) = (0, 0)$.

We endow the set $C(X)$ with a Banach space structure using the norm:

$$(11.4) \quad \|\varphi\| = \sup_{(x, v) \in X} |\varphi(x, v)|$$

We need to impose suitable regularity and compatibility conditions in the class of test functions in order to take into account the compatibility conditions imposed in X . Let V be an open subset of \mathcal{U} . We will consider functions $\zeta \in C(\bar{V})$ satisfying:

$$(11.5) \quad \zeta(0, -v) = r^2 \zeta(0, rv) \ ; \ v > 0 \ , \ \text{if } (0, -v), (0, rv) \in \bar{V}$$

$$(11.6) \quad \text{There exist } \zeta_x, \zeta_{vv} \in C(\bar{V})$$

$$(11.7) \quad \zeta_x(0, -v) = r^2 \zeta_x(0, rv) \ ; \ v > 0 \ \text{if } (0, -v), (0, rv) \in \bar{V}$$

$$(11.8) \quad \operatorname{supp}(\zeta) \cap [\{x + |v| \geq R\} \cup \{0, 0\}] = \emptyset \ \text{for some } R > 0$$

We can define a set of functions

$$(11.9) \quad \mathcal{F}(V) = \{\zeta \in C(\bar{V}) : (11.5), (11.6), (11.7), (11.8) \text{ hold}\}$$

We now define the action of the operator \mathcal{L} in a subset of $C(X)$.

Definition 11.7. Suppose that W is any open subset of X . Given $\varphi \in C(W)$, we will say that $\mathcal{L}\varphi$ is defined if there exists $w \in C(W)$ such that for any $\zeta \in \mathcal{F}(\mathcal{U} \cap W)$ we have:

$$(11.10) \quad \int_{\mathcal{U} \cap W} \varphi \mathcal{L}^*(\zeta) dx dv = \int_{\mathcal{U} \cap W} w \zeta dx dv$$

where $\mathcal{L}^* = D_v^2 - v D_x$. We will then write $w = \mathcal{L}\varphi$. Given $V \subset \mathcal{U}$, we will say that $\mathcal{L}\varphi$ is defined in V if there exists $w \in C(V)$ satisfying (11.1) such that for any $\zeta \in \mathcal{F}(\mathcal{U})$ such that $\text{supp}(\zeta) \cap (\partial V) = \emptyset$, (11.10) holds.

12. REGULARITY PROPERTIES OF THE SOLUTIONS: HYPOELLIPTICITY.

In this Section we will formulate some regularity results associated to PDEs containing the operator \mathcal{L} which will be used repeatedly in the following.

One of the key features of the solutions of equations like (1.3) is that their solutions are smooth in spite of the fact that they contain only second derivatives in the direction of v and not in the direction of x . This smoothness can be proved for a large class of initial data using the fundamental solution associated to this equation in the whole space, which was first computed by Kolmogorov (cf. [37]).

On the other hand, hypoellipticity properties for equations with the form (1.3) have been known for a long time and they have been formulated in different functional spaces in several papers (cf. [5], [25], [26], [40], [47], [49]). Hypoellipticity properties for the evolution problem associated to the equation (1.3) with absorbing boundary conditions have been proved in [31]. In this paper, we will need regularizing effects only for the stationary problem. We collect in this Section some regularity results used in this paper. We first need some notation to denote a portion of the boundaries of a class of domains which will be used repeatedly in this paper. More precisely, we will restrict ourselves to the following class of domains contained in $[0, \infty) \times (-\infty, \infty) \setminus \{(0, 0)\}$:

Definition 12.1. $\Xi \subset [0, \infty) \times (-\infty, \infty) \setminus \{(0, 0)\}$ is an admissible domain if it has one of the following forms:

(a) It is a cartesian product $(x_1, x_2) \times (v_1, v_2)$ with $(x_1, x_2) \subset (0, \infty)$, $(v_1, v_2) \subset \mathbb{R}$ and $0 \notin [x_1, x_2] \times [v_1, v_2]$.

(b) It has the form $(0, x_2) \times (v_1, v_2) \setminus (0, x_1) \times (\bar{v}_1, \bar{v}_2)$ with $0 < x_1 < x_2$, $v_1 < \bar{v}_1 < 0 < \bar{v}_2 < v_2$.

In the case (a), we will denote as $\partial_a \Xi$ and term as admissible boundary the subset of the boundary $\partial \Xi$ defined by means of:

$$\begin{aligned} \partial_a \Xi &= \partial_{a,h} \Xi \cup \partial_{a,v} \Xi \\ \partial_{a,h} \Xi &= ([x_1, x_2] \times \{v_1\}) \cup ([x_1, x_2] \times \{v_2\}) \\ \partial_{a,v} \Xi &= (([v_1, v_2] \cap (0, \infty)) \times \{x_2\}) \cup (([v_1, v_2] \cap (-\infty, 0)) \times \{x_1\}). \end{aligned}$$

and we will denote as $\partial_a^* \Xi$ the adjoint admissible boundary given by:

$$\begin{aligned} \partial_a^* \Xi &= \partial_{a,h} \Xi \cup \partial_{a,v}^* \Xi \\ \partial_{a,v}^* \Xi &= (([v_1, v_2] \cap (-\infty, 0)) \times \{x_2\}) \cup (([v_1, v_2] \cap (0, \infty)) \times \{x_1\}). \end{aligned}$$

In the case (b) we will denote as $\partial_a \Xi$ the set:

$$\begin{aligned} \partial_a \Xi &= \partial_{a,h} \Xi \cup \partial_{a,v} \Xi \\ \partial_{a,h} \Xi &= ([0, x_2] \times \{v_1\}) \cup ([0, x_2] \times \{v_2\}) \cup ([0, x_1] \times \{\bar{v}_1\}) \cup ([0, x_1] \times \{\bar{v}_2\}) \\ \partial_{a,v} \Xi &= ([\bar{v}_1, 0] \times \{x_1\}) \cup ([0, v_2] \times \{x_2\}) \end{aligned}$$

and the adjoint admissible boundary $\partial_a^* \Xi$ as:

$$\begin{aligned}\partial_a^* \Xi &= \partial_{a,h} \Xi \cup \partial_{a,v}^* \Xi \\ \partial_{a,v}^* \Xi &= ([0, v_2] \times \{x_2\}) \cup ([\bar{v}_1, 0] \times \{x_1\}).\end{aligned}$$

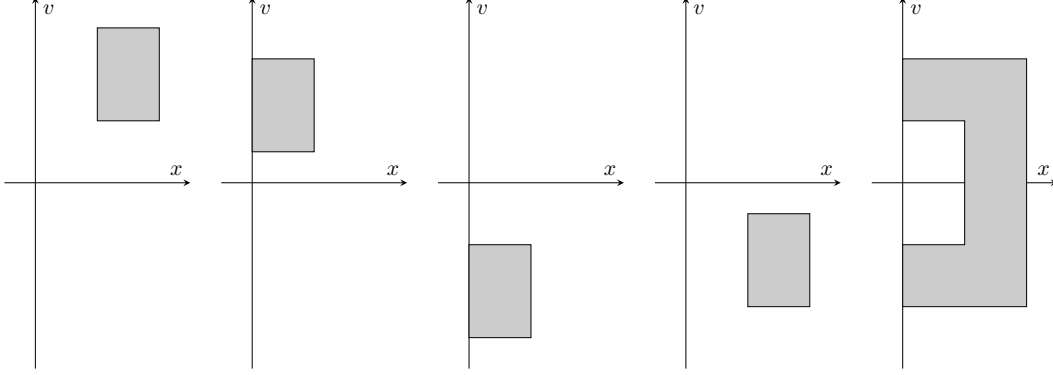


FIGURE 4. Admissible domains in Definition 12.1

Theorem 12.2. *Let $W \subset X$ be one admissible domain in the sense of Definition 12.1 and $\varphi \in C(W)$. Suppose that $\mathcal{L}\varphi = w \in C(W)$ in the sense of Definition 11.7. Then $D_v\varphi \in C(W)$. Moreover, if W_δ denotes the subset of W such that $\text{dist}((x, v), \partial_a W) \geq \delta > 0$ we have:*

$$\|D_v\varphi\|_{L^\infty(W_\delta)} \leq C_{\delta, W} \|\varphi\|_{L^\infty(W)}$$

where $C_{\delta, W} > 0$ depends only on W, δ .

Proof. The regularity at the interior points of $\mathcal{U} \cap W$ is a consequence of the hypoellipticity results in [25]. The regularity in the sets $\{x = 0, v > 0\} \cap W$ follows from classical parabolic theory, assuming that x is the time variable. We then obtain the desired continuity in $\{x = 0, v < 0\} \cap W$ using (11.1). The uniform regularity in $\{v < 0\} \cap W$ then follows from parabolic theory. \square

We will use also the following result for the solutions of the evolution problem (1.3), (1.4).

Theorem 12.3. *Let $W = \mathcal{R}_2 \setminus \overline{\mathcal{R}_{\frac{1}{2}}}$ with \mathcal{R}_δ as in Definition 4.7. Suppose that $P, P_t, P_{xx}, P_v \in C([0, 1] \times W)$ and that P solves (1.3), (1.4) in $(0, 1) \times W$. Then:*

$$\|D_v P\|_{L^\infty\left(\left[\frac{1}{2}, 1\right] \times \left(\mathcal{R}_{\frac{3}{2}} \setminus \overline{\mathcal{R}_{\frac{2}{3}}}\right)\right)} \leq C \|P\|_{L^\infty([0, 1] \times W)}$$

Proof. It is just a consequence of the results in [25], [49] as well as the boundary condition (1.4). \square

We will use also the following hypoellipticity property.

Theorem 12.4. *Suppose that $W \subset X$ and $\varphi \in C(W)$, with $(0, 0) \notin W$.*

(i) *Suppose that $\mathcal{L}\varphi = w \in C(W)$ in the sense of Definition 11.7. Then $D_x\varphi, D_v\varphi, D_v^2\varphi \in L_{loc}^p(W)$ and these norms can be estimated by $\|w\|_\infty$.*

(ii) *Suppose that $\mathcal{L}\varphi = \nu \in \mathcal{M}(W)$. Then the L^p norm of φ and $D_v\varphi$ in each open set, or in any horizontal curve can be estimated by the sum of the L^1 norm of φ and the $\mathcal{M}(W)$ norm of ν .*

Proof. The result (i) follows from [49]. The result (ii) can be proved by duality, using (i) for the adjoint operator of \mathcal{L} . \square

... I think that the case of horizontal curves follows because we have enough smoothing effects for one derivative. Perhaps check this. ...

13. DEFINITION OF THE OPERATORS Ω_σ . DOMAINS $\mathcal{D}(\Omega_\sigma)$.

We now define some operators Ω_σ for the different boundary conditions described in Section 16 of Section 3, where σ is the subindex labelling each set of boundary conditions. In all the cases the operator Ω_σ acts over continuous functions defined on a compact topological space X in Definition 11.1.

We will assume that the functions $\varphi \in \mathcal{D}(\Omega_\sigma)$ have the following asymptotic behaviour near the singular set:

$$(13.1) \quad \varphi(x, v) = \varphi(0, 0) + \mathcal{A}(\varphi) F_\beta(x, v) + \psi(x, v), \quad \lim_{R \rightarrow 0} \frac{\sup_{x+|v|^3=R} |\psi(x, v)|}{R^\beta} = 0$$

where $\mathcal{A}(\varphi) \in \mathbb{R}$ and with F_β as in (10.7).

13.1. The case $r < r_c$. Trapping boundary conditions. In this case we will define $\Omega_{t,sub}\varphi$ as follows. We consider the domain:

$$(13.2) \quad \mathcal{D}(\Omega_{t,sub}) = \{\varphi, \mathcal{L}\varphi \in C(X) : \varphi \text{ satisfies (13.1), } \lim_{(x,v) \rightarrow (0,0)} (\mathcal{L}\varphi)(x, v) = 0\}$$

We then define:

$$(13.3) \quad \begin{aligned} (\Omega_{t,sub}\varphi)(x, v) &= (\mathcal{L}\varphi)(x, v), \quad \text{if } (x, v) \neq (0, 0), \infty \\ (\Omega_{t,sub}\varphi)(0, 0) &= \lim_{(x,v) \rightarrow (0,0)} (\mathcal{L}\varphi)(x, v) = 0, \quad (\Omega_{t,sub}\varphi)(\infty) = (\mathcal{L}\varphi)(\infty) \end{aligned}$$

with $\mathcal{L}\varphi$ as in Definition 11.7. We remark that in this case as well as in the following three cases, we have that $\mathcal{L}\varphi \in C(X)$ for the functions φ in the domains and then the limit $\lim_{(x,v) \rightarrow (0,0)} (\mathcal{L}\varphi)(x, v)$ exists.

13.2. The case $r < r_c$. Nontrapping boundary conditions. We define $\Omega_{nt,sub}\varphi$ by means of (11.3) in the domain:

$$(13.4) \quad \mathcal{D}(\Omega_{nt,sub}) = \{\varphi, \mathcal{L}\varphi \in C(X) : \varphi \text{ satisfies (13.1) and } \mathcal{A}(\varphi) = 0\}$$

We then define:

$$(13.5) \quad \begin{aligned} (\Omega_{nt,sub}\varphi)(x, v) &= (\mathcal{L}\varphi)(x, v), \quad \text{if } (x, v) \neq (0, 0), \infty \\ (\Omega_{nt,sub}\varphi)(0, 0) &= \lim_{(x,v) \rightarrow (0,0)} (\mathcal{L}\varphi)(x, v), \quad (\Omega_{nt,sub}\varphi)(\infty) = (\mathcal{L}\varphi)(\infty) \end{aligned}$$

with $\mathcal{L}\varphi$ as in Definition 11.7.

13.3. The case $r < r_c$. Partially trapping boundary conditions. Given any $\mu_* > 0$, we define $\Omega_{pt,sub}\varphi$ by means of (11.3) in the domain:

$$(13.6) \quad \mathcal{D}(\Omega_{pt,sub}) = \left\{ \varphi, \mathcal{L}\varphi \in C(X) : \varphi \text{ satisfies (13.1), there exists } \lim_{(x,v) \rightarrow (0,0)} (\mathcal{L}\varphi)(x, v) = \mu_* |C_*| \mathcal{A}(\varphi) \right\}$$

where C_* is as in (10.11) (cf. also Proposition 10.3). We will denote from now on $\lim_{(x,v) \rightarrow (0,0)} (\mathcal{L}\varphi)(x, v) = (\mathcal{L}\varphi)(0, 0)$. We then define:

$$(13.7) \quad \begin{aligned} (\Omega_{pt,sub}\varphi)(x, v) &= (\mathcal{L}\varphi)(x, v), \quad \text{if } (x, v) \neq (0, 0), \infty \\ (\Omega_{pt,sub}\varphi)(0, 0) &= (\mathcal{L}\varphi)(0, 0) = \mu_* |C_*| \mathcal{A}(\varphi), \quad (\Omega_{pt,sub}\varphi)(\infty) = (\mathcal{L}\varphi)(\infty) \end{aligned}$$

We will not make explicit the dependence of the operators $\Omega_{pt,sub}$ in μ_* for the sake of simplicity. Notice that trapping boundary conditions reduce formally to the case $\mu_* = 0$ and nontrapping boundary conditions to the case $\mu_* = \infty$.

13.4. The case $r > r_c$. In this case we only consider the case of nontrapping boundary conditions. We then define $\Omega_{sup}\varphi$ by means of (11.3) in the domain:

$$(13.8) \quad \mathcal{D}(\Omega_{sup}) = \left\{ \varphi, \mathcal{L}\varphi \in C(X) : \text{there exists } \lim_{(x,v) \rightarrow (0,0)} (\mathcal{L}\varphi)(x,v) \right\}$$

Notice that in this case the condition (13.1) does not make sense if $\varphi \in C(X)$ because $\beta < 0$.

We then define

$$(13.9) \quad \begin{aligned} (\Omega_{sup}\varphi)(x,v) &= (\mathcal{L}\varphi)(x,v), \text{ if } (x,v) \neq (0,0), \infty \\ (\Omega_{sup}\varphi)(0,0) &= \lim_{(x,v) \rightarrow (0,0)} (\mathcal{L}\varphi)(x,v), \quad (\Omega_{sup}\varphi)(\infty) = (\mathcal{L}\varphi)(\infty) \end{aligned}$$

14. FORMULATION OF THE ADJOINT PROBLEMS IF $r < r_c$.

In this Section we prove the following characterizations of the adjoint operators \mathbb{A} for the boundary conditions (5.8), (5.9), (5.11).

Proposition 14.1. *Let $r < r_c$. The operator $\Omega_{t,sub}$ defined in Section 13.1 is an adjoint operator \mathbb{A} for the problem (1.3), (1.4) with boundary conditions (5.9) in the sense of Definition 8.1.*

Proposition 14.2. *Let $r < r_c$. The operator $\Omega_{nt,sub}$ defined in Section 13.2 is an adjoint operator \mathbb{A} for the problem (1.3), (1.4) with boundary conditions (5.8) in the sense of Definition 8.1.*

Proposition 14.3. *Let $r < r_c$. The operator $\Omega_{pt,sub}$ defined in Section 13.3 is an adjoint operator \mathbb{A} for the problem (1.3), (1.4) with boundary conditions (5.11) in the sense of Definition 8.1.*

Proof of Propositions 14.1, 14.2, 14.3. We use Definition 8.1. We will assume in the following that P is a smooth function outside the singular point satisfying (1.3), (1.4) and (5.1). Suppose first that P satisfies also (5.9) and that $\varphi(\cdot, t) \in \mathcal{D}(\Omega_{t,sub})$ for all $t \in [0, T]$. We then compute the integral:

$$(14.1) \quad \mathcal{I} = \int_{[0,T]} \int_{\mathcal{U}} P(\partial_t \varphi + \Omega_{t,sub}\varphi) dx dv dt$$

Notice that the assumptions on φ in Definition 8.1 as well as the asymptotics (5.1) imply that $|P| |\partial_t \varphi|$ and $|P| |\Omega_{t,sub}\varphi|$ belong to $L^1([0, T] \times \mathbb{R}_+^2)$. Therefore:

$$(14.2) \quad \begin{aligned} \mathcal{I} &= \lim_{\delta \rightarrow 0} \int_{[0,T]} \int_{\mathcal{U} \setminus \mathcal{R}_\delta} P(\partial_t \varphi + \Omega_{t,sub}\varphi) dx dv dt \\ &= \int_{[0,T]} \left(\lim_{\delta \rightarrow 0} \int_{\mathcal{U} \setminus \mathcal{R}_\delta} P(\partial_t \varphi + \Omega_{t,sub}\varphi) dx dv \right) dt \end{aligned}$$

with \mathcal{R}_δ as in (4.27) with $b = 1$. Notice that $\Omega_{t,sub}\varphi = \mathcal{L}\varphi$ for $(x, v) \neq (0, 0)$. Theorem 12.2 implies that $D_v\varphi$ is continuous outside the singular point $(x, v) = (0, 0)$. Then:

$$(14.3) \quad \int_{\mathcal{U} \setminus \mathcal{R}_\delta} P \Omega_{t,sub}\varphi dx dv = \mathcal{J}_\delta + \int_{\mathcal{U} \setminus \mathcal{R}_\delta} \varphi (D_v^2 P - v D_x P) dx dv$$

with

$$(14.4) \quad \mathcal{J}_\delta(t) = \int_{\partial\mathcal{R}_\delta} [P(n_v D_v \varphi + n_x v \varphi) - \varphi D_v P n_v] ds$$

where ds is the arc-length of $\partial\mathcal{R}_\delta$ and the normal vector is pointing towards \mathcal{R}_δ . Notice that in the derivation of this formula we have used the existence of the derivative $D_v P$. On the other hand, using that $|P| |\partial_t \varphi| \in L^1([0, T] \times \mathbb{R}_+^2)$ we obtain:

$$(14.5) \quad \begin{aligned} \int_{[0, T]} \int_{\mathcal{U} \setminus \mathcal{R}_\delta} P \partial_t \varphi dx dv dt &= \int_{\mathcal{U} \setminus \mathcal{R}_\delta} (P \varphi)(\cdot, T) dx dv - \int_{\mathcal{U} \setminus \mathcal{R}_\delta} (P \varphi)(\cdot, 0) dx dv - \\ &\quad - \int_{[0, T]} \int_{\mathcal{U} \setminus \mathcal{R}_\delta} \varphi \partial_t P dx dv dt \end{aligned}$$

Combining (14.2), (14.3), (14.5) and using also (1.3), (1.4) we obtain:

$$\mathcal{I} = \lim_{\delta \rightarrow 0} \int_{[0, T]} \mathcal{J}_\delta(t) dt + \lim_{\delta \rightarrow 0} \int_{\mathcal{U} \setminus \mathcal{R}_\delta} (P \varphi)(\cdot, T) dx dv - \lim_{\delta \rightarrow 0} \int_{\mathcal{U} \setminus \mathcal{R}_\delta} (P \varphi)(\cdot, 0) dx dv$$

whence:

$$(14.6) \quad \mathcal{I} = \lim_{\delta \rightarrow 0} \int_{[0, T]} \mathcal{J}_\delta(t) dt + \int_{\mathcal{U}} (P \varphi)(\cdot, T) dx dv - \int_{\mathcal{U}} (P \varphi)(\cdot, 0) dx dv$$

We compute $\lim_{\delta \rightarrow 0} \int_{[0, T]} \mathcal{J}_\delta(t) dt$ as follows. Using the asymptotics (5.1), (13.1) as well as (5.9) we can write:

$$(14.7) \quad \begin{aligned} \mathcal{J}_\delta(t) &= a_{-\frac{2}{3}}(t) \varphi(0, 0, t) \int_{\partial\mathcal{R}_\delta} [(n_x v) G_{-\frac{2}{3}} - n_v D_v G_{-\frac{2}{3}}] ds + \\ &\quad + \varphi(0, 0, t) \int_{\partial\mathcal{R}_\delta} [n_x v Q - D_v Q n_v] ds + \\ &\quad + \int_{\partial\mathcal{R}_\delta} [P(n_v D_v W + n_x v W) - W D_v P n_v] ds \end{aligned}$$

where $Q = P - a_{-\frac{2}{3}}(t) G_{-\frac{2}{3}}$ and $W = \varphi - \varphi(0, 0, t)$. We define functions $Q_R(x, v) = \frac{Q(Rx, R^{\frac{1}{3}}v)}{R^{-\frac{2}{3}}}$, $W_R(x, v) = \frac{W(Rx, R^{\frac{1}{3}}v)}{R^\beta}$. Using (5.1) and (13.1) we obtain that $|Q_R(x, v)|$ and $|W_R(x, v)|$ are bounded for $\mathcal{R}_2 \setminus \mathcal{R}_{\frac{1}{2}}$. Theorem 12.3 combined with (5.1) then imply that $\sup_{\mathcal{R}_{\frac{3}{2}} \setminus \mathcal{R}_{\frac{2}{3}}} |D_v Q_R| \rightarrow 0$ as $R \rightarrow 0$. On the other hand, Theorem 12.2 and (13.1) yield $\sup_{\mathcal{R}_{\frac{3}{2}} \setminus \mathcal{R}_{\frac{2}{3}}} |D_v W_R| \leq h(t)$, with $\int_{[0, T]} h(t) dt < \infty$. The definition of Q_R , W_R then yields:

$$(14.8) \quad \lim_{R \rightarrow 0} \sup_{\frac{R}{2} \leq x+|v|^3 \leq 2R} \frac{|D_v Q|}{R^{-\frac{2}{3}-\frac{1}{3}}} = 0, \quad \int_{[0, T]} \left[\sup_{\frac{R}{2} \leq x+|v|^3 \leq 2R} (|W| + R^{\frac{1}{3}} |D_v W|) \right] dt \leq C R^\beta$$

Therefore, the last two integral terms in (14.7) converge to zero (after integrating in time) and we have:

$$\lim_{\delta \rightarrow 0} \int_{[0, T]} \mathcal{J}_\delta(t) dt = \left(\int_{[0, T]} a_{-\frac{2}{3}}(t) \varphi(0, 0, t) dt \right) \lim_{\delta \rightarrow 0} \int_{\partial\mathcal{R}_\delta} [(n_x v) G_{-\frac{2}{3}} - n_v D_v G_{-\frac{2}{3}}] ds$$

The last limit can be computed using Proposition 4.8. Then:

$$\lim_{\delta \rightarrow 0} \int_{[0, T]} \mathcal{J}_\delta(t) dt = -9^{\frac{2}{3}} \left[\log(r) + \frac{\pi}{\sqrt{3}} \right] \left(\int_{[0, T]} a_{-\frac{2}{3}}(t) \varphi(0, 0, t) dt \right)$$

We then obtain, using (14.6):

$$\begin{aligned} \mathcal{I} = & -9^{\frac{2}{3}} \left[\log(r) + \frac{\pi}{\sqrt{3}} \right] \left(\int_{[0,T]} a_{-\frac{2}{3}}(t) \varphi(0,0,t) dt \right) \\ & + \int_{\mathcal{U}} (P\varphi)(\cdot, T) dx dv - \int_{\mathcal{U}} (P\varphi)(\cdot, 0) dx dv \end{aligned}$$

Using now (5.7) and (7.2) we obtain:

$$\mathcal{I} = \left(\int_{[0,T]} \frac{dm(t)}{dt} \varphi(0,0,t) dt \right) + \int_{\mathcal{U}} (P\varphi)(\cdot, T) dx dv - \int_{\mathcal{U}} (P\varphi)(\cdot, 0) dx dv$$

whence, using (7.1) and (14.1) we obtain (8.1). Notice that we use also that $\varphi_t(0,0,t) = -\mathcal{L}\varphi(0,0,t) = 0$ due to (13.2). This concludes the Proof of Proposition 14.1.

We now consider the case of nontrapping boundary conditions (cf. Proposition 14.2). Arguing similarly we obtain formula (14.6) with:

$$\mathcal{I} = \int_{[0,T]} \int_{\mathcal{U}} P(\partial_t \varphi + \Omega_{nt,sub} \varphi) dx dv dt$$

In order to compute $\lim_{\delta \rightarrow 0} \int_{[0,T]} \mathcal{J}_{\delta}(t) dt$ in this case we use (5.1), (13.1) and (5.8). We then obtain, instead of (14.7):

$$\begin{aligned} \mathcal{J}_{\delta}(t) = & a_{\alpha}(t) \varphi(0,0,t) \int_{\partial \mathcal{R}_{\delta}} [(n_x v) G_{\alpha} - n_v D_v G_{\alpha}] ds + \\ & + \varphi(0,0,t) \int_{\partial \mathcal{R}_{\delta}} [n_x v Q - D_v Q n_v] ds + \\ (14.9) \quad & + \int_{\partial \mathcal{R}_{\delta}} [P(n_v D_v W + n_x v W) - W D_v P n_v] ds \end{aligned}$$

where now $Q = P - a_{\alpha}(t) G_{\alpha}$ and $W = \varphi - \varphi(0,0,t)$. Notice that the boundary condition $\mathcal{A}(\varphi) = 0$ in (13.4) implies, arguing as in the Proof of (14.8):

$$(14.10) \quad \lim_{R \rightarrow 0} \sup_{\frac{R}{2} \leq x+|v|^3 \leq 2R} \left(\frac{|Q|}{R^{-\frac{2}{3}}} + \frac{|D_v Q|}{R^{-\frac{2}{3}-\frac{1}{3}}} \right) = 0, \quad \lim_{R \rightarrow 0} \int_{[0,T]} \left[\sup_{\frac{R}{2} \leq x+|v|^3 \leq 2R} \left(\frac{W}{R^{\beta}} + \frac{|D_v W|}{R^{\beta-\frac{1}{3}}} \right) \right] dt = 0$$

Then, the two last integrals in (14.9) tend to zero. On the other hand, the remaining one vanishes due to Proposition 4.11. Therefore $\lim_{\delta \rightarrow 0} \int_{[0,T]} \mathcal{J}_{\delta}(t) dt = 0$. Using then (5.8), (5.7) and (7.2) it follows that $m(t) = 0$, whence (8.1) follows. This shows Proposition 14.2.

Finally we consider the case of Partially Trapping Boundary Conditions (cf. Proposition 14.3). In this case we obtain (14.6) with:

$$\mathcal{I} = \int_{[0,T]} \int_{\mathcal{U}} P(\partial_t \varphi + \Omega_{pt,sub} \varphi) dx dv dt$$

We now have:

$$\begin{aligned}
\mathcal{J}_\delta(t) &= a_\alpha(t) \varphi(0, 0, t) \int_{\partial\mathcal{R}_\delta} [(n_x v) G_\alpha - n D_v G_\alpha] ds + \\
&\quad + a_{-\frac{2}{3}}(t) \varphi(0, 0, t) \int_{\partial\mathcal{R}_\delta} \left[(n_x v) G_{-\frac{2}{3}} - n D_v G_{-\frac{2}{3}} \right] ds \\
&\quad + \varphi(0, 0, t) \int_{\partial\mathcal{R}_\delta} [n_x v Q - D_v Q n_v] ds + \\
(14.11) \quad &\quad + \int_{\partial\mathcal{R}_\delta} [P(n_v D_v W + n_x v W) - W D_v P n_v] ds
\end{aligned}$$

with $Q = P - a_\alpha(t) G_\alpha - a_{-\frac{2}{3}}(t) G_{-\frac{2}{3}}$, $W = \varphi - \varphi(0, 0, t)$. Arguing as in the previous cases, by means of a rescaling argument, we obtain:

$$\begin{aligned}
(14.12) \quad &\lim_{R \rightarrow 0} \sup_{\frac{R}{2} \leq x+|v|^3 \leq 2R} \left(\frac{|Q|}{R^{-\frac{2}{3}}} + \frac{|D_v Q|}{R^{-\frac{2}{3}-\frac{1}{3}}} \right) = 0 \\
&\lim_{R \rightarrow 0} \sup_{\frac{R}{2} \leq x+|v|^3 \leq 2R} \int_{[0, T]} \left(\frac{W}{R^\beta} + \frac{|D_v W|}{R^{\beta-\frac{1}{3}}} \right) \leq C
\end{aligned}$$

The first integral on the right-hand side of (14.11) vanishes due to Proposition 4.11. The third integral tends to zero as $\delta \rightarrow 0$ due to (14.12). Using Proposition 4.8 we obtain:

$$\begin{aligned}
(14.13) \quad \lim_{\delta \rightarrow 0} \mathcal{J}_\delta(t) &= -9^{\frac{2}{3}} \left[\log(r) + \frac{\pi}{\sqrt{3}} \right] a_{-\frac{2}{3}}(t) \varphi(0, 0, t) + \\
&\quad + \lim_{\delta \rightarrow 0} \int_{\partial\mathcal{R}_\delta} [P(n_v D_v W + n_x v W) - W D_v P n_v] ds
\end{aligned}$$

We now notice that:

$$(14.14) \quad \lim_{\delta \rightarrow 0} \int_{\partial\mathcal{R}_\delta} [P(n_v D_v W + n_x v W) - W D_v P n_v] ds = C_* a_\alpha(t) \mathcal{A}(\varphi)$$

where C_* is as in (10.11). We recall that C_* has been computed in Proposition 10.3. Therefore, using (14.14) in (14.13) we obtain:

$$\lim_{\delta \rightarrow 0} \mathcal{J}_\delta(t) = -9^{\frac{2}{3}} \left[\log(r) + \frac{\pi}{\sqrt{3}} \right] a_{-\frac{2}{3}}(t) \varphi(0, 0, t) + C_* a_\alpha(t) \mathcal{A}(\varphi)$$

Combining (5.7), (7.2), (14.6), (14.13) we obtain:

$$\begin{aligned}
\mathcal{I} &= \int_{[0, T]} \frac{dm(t)}{dt} \varphi(0, 0, t) dt + C_* \int_{[0, T]} a_\alpha(t) \mathcal{A}(\varphi) dt \\
&\quad + \int_{\mathcal{U}} (P\varphi)(\cdot, T) dx dv - \int_{\mathcal{U}} (P\varphi)(\cdot, 0) dx dv
\end{aligned}$$

Using then the definition of the measure f in (7.1) as well as the fact that $\partial_t \varphi(0, 0, t) = -\mathcal{L}\varphi(0, 0, t)$ we obtain:

$$\mathcal{I} = \int_{[0, T]} m(t) \mathcal{L}\varphi(0, 0, t) dt + C_* \int_{[0, T]} a_\alpha(t) \mathcal{A}(\varphi) dt + \int_{\mathcal{V}} f(dx dv, T) - \int_{\mathcal{V}} f(dx dv, 0)$$

Using now (5.10) and (13.6) we obtain:

$$\begin{aligned}\mathcal{I} &= -\mu_* C_* \int_{[0,T]} m(t) \mathcal{A}(\varphi) dt + \mu_* C_* \int_{[0,T]} m(t) \mathcal{A}(\varphi) dt \\ &\quad + \int_{\mathcal{V}} f(dx dv, T) - \int_{\mathcal{V}} f(dx dv, 0) \\ &= \int_{\mathcal{V}} f(dx dv, T) - \int_{\mathcal{V}} f(dx dv, 0)\end{aligned}$$

where we used the fact that $C_* < 0$ (cf. Lemma 10.4). Then, using also the definition of \mathcal{I} in (14.2) we obtain :

$$\int_{[0,T]} \int_{\mathcal{U}} P(\partial_t \varphi + \Omega_{a,m} \varphi) dx dv dt = \int_{\mathcal{V}} f(dx dv, T) - \int_{\mathcal{V}} f(dx dv, 0)$$

whence (8.1) follows. This concludes the Proof of Proposition 14.3. \square

15. FORMULATION OF THE ADJOINT PROBLEM IF $r > r_c$.

Proposition 15.1. *Let $r > r_c$. The operator Ω_{sup} defined in Section 13.4 is an adjoint operator \mathbb{A} for the problem (1.3), (1.4) with boundary conditions (6.1) in the sense of Definition 8.1.*

Proof. It is similar to the proof of Propositions 14.1, 14.2, 14.3. We define

$$(15.1) \quad \mathcal{I} = \int_{[0,T]} \int_{\mathcal{U}} P(\partial_t \varphi + \Omega_{\text{sup}} \varphi) dx dv dt$$

By assumption P satisfies (5.1) with (6.1). We define $Q = P - a_\alpha(t) G_\alpha$ and $W = \varphi - \varphi(0, 0, t)$. Then, arguing as in the previous proof we obtain (14.6) with:

$$\begin{aligned}(15.2) \quad \mathcal{J}_\delta(t) &= a_\alpha(t) \varphi(0, 0, t) \int_{\partial \mathcal{R}_\delta} [(n_x v) G_\alpha - n_v D_v G_\alpha] ds + \\ &\quad + \varphi(0, 0, t) \int_{\partial \mathcal{R}_\delta} [n_x v Q - D_v Q n_v] ds + \\ &\quad + \int_{\partial \mathcal{R}_\delta} [P(n_v D_v W + n_x v W) - W D_v P n_v] ds\end{aligned}$$

Notice that in this case we have $\alpha > -\frac{2}{3}$ and due to (6.1) we have, arguing as in the previous case:

$$(15.3) \quad \lim_{R \rightarrow 0} \sup_{\frac{R}{2} \leq x+|v|^3 \leq 2R} \left(\frac{|Q|}{R^{-\frac{2}{3}}} + \frac{|D_v Q|}{R^{-\frac{2}{3}-\frac{1}{3}}} \right) = 0, \quad \lim_{R \rightarrow 0} \int_{[0,T]} \left[\sup_{\frac{R}{2} \leq x+|v|^3 \leq 2R} \left(|W| + \frac{|D_v W|}{R^{-\frac{1}{3}}} \right) \right] dt = 0$$

Notice that, since $\beta < 0$ in this case, the only information that we have about $|W|$ near the origin is that it converges to zero, plus the estimates for the derivatives that can be obtained by rescaling. The first integral on the right of (15.2) vanishes due to Proposition 4.11. The second converges to zero as $\delta \rightarrow 0$ due to (15.3) and the third one can be estimated, using the estimates for P as well as (15.3) as $C(\delta)^{\alpha+\frac{2}{3}}$. Using that $\alpha > -\frac{2}{3}$ it then follows that the this integral converges to zero as $\delta \rightarrow 0$. Taking the limit of (14.6) as $\delta \rightarrow 0$ we arrive at:

$$\mathcal{I} = \int_{\mathcal{U}} (P\varphi)(\cdot, T) dx dv - \int_{\mathcal{U}} (P\varphi)(\cdot, 0) dx dv$$

Using then that $m(t) = 0$ (cf. (5.7) and (7.1)) we obtain (8.1) and the result follows. \square

16. SUMMARY OF ADJOINT PROBLEMS TO BE CONSIDERED.

Given the problem (1.3), (1.4) with any of the boundary conditions (5.8), (5.9), (5.11), (6.1) we define the adjoint problem as:

$$(16.1) \quad \varphi_t + \mathbb{A}(\varphi) = 0, \quad t \in (0, T), \quad \varphi(\cdot, T) = \varphi_0(\cdot)$$

where $\varphi(\cdot, t) \in \mathcal{D}(\mathbb{A})$ for any $t \in (0, T)$. We change the time as $t \rightarrow (T - t)$ in order to obtain a forward parabolic problem. Therefore (16.1) becomes:

$$(16.2) \quad \varphi_t - \mathbb{A}(\varphi) = 0, \quad t \in (0, T), \quad \varphi(\cdot, 0) = \varphi_0(\cdot)$$

with $\varphi(\cdot, t) \in \mathcal{D}(\mathbb{A})$ for any $t \in (0, T)$. We will say that (16.2) is the adjoint problem of (1.3), (1.4) with the corresponding boundary condition (5.8), (5.9), (5.11), (6.1). Notice that the characterizations of the adjoint operators \mathbb{A} obtained in Propositions 15.1, 14.1, 14.2, 14.3 make possible to reformulate the problem (16.2) as a PDE problem with suitable boundary conditions along the line $L^* = \{(x, v) = (0, v) : v \in \mathbb{R}\}$ as well as near the singular point $(x, v) = (0, 0)$. We now describe in detail this set of PDE problems for the different cases for further reference.

16.1. The case $r < r_c$.: In this case we need to distinguish the cases of the three boundary conditions (5.8), (5.9), (5.11). Using the definitions of the operators $\Omega_{t,sub}$, $\Omega_{nt,sub}$, $\Omega_{pt,sub}$ in Sections 13.1, 13.2, 13.3 as well as Propositions 14.1, 14.2, 14.3 we obtain the following:

The adjoint problem of (1.3), (1.4) with boundary condition (5.8) is:

$$(16.3) \quad \partial_t \varphi - v \partial_x \varphi - \partial_{vv} \varphi = 0, \quad x > 0, \quad v \in \mathbb{R}$$

$$(16.4) \quad \varphi(0, -v, t) = \varphi(0, rv, t), \quad v > 0$$

$$(16.5) \quad \varphi(x, v, t) = \varphi(0, 0, t) + \psi(x, v, t), \quad \lim_{R \rightarrow 0} \frac{\sup_{x+|v|^3=R} |\psi(x, v, t)|}{R^\beta} = 0, \quad t \in (0, T)$$

which must be complemented with the initial condition:

$$(16.6) \quad \varphi(x, v, 0) = \varphi_0(x, v)$$

The adjoint problem of (1.3), (1.4) with boundary condition (5.9) is:

$$(16.7) \quad \partial_t \varphi - v \partial_x \varphi - \partial_{vv} \varphi = 0, \quad x > 0, \quad v \in \mathbb{R}$$

$$(16.8) \quad \varphi(0, -v, t) = \varphi(0, rv, t), \quad v > 0$$

$$(16.9) \quad \lim_{(x,v) \rightarrow (0,0)} (\mathcal{L}\varphi)(x, v, t) = 0, \quad t \in (0, T)$$

with \mathcal{L} as in (11.3). A natural initial condition for this problem is:

$$(16.10) \quad \varphi(x, v, 0) = \varphi_0(x, v)$$

The adjoint problem of (1.3), (1.4) with boundary condition (5.11) is:

$$(16.11) \quad \partial_t \varphi - v \partial_x \varphi - \partial_{vv} \varphi = 0, \quad x > 0, \quad v \in \mathbb{R}$$

$$(16.12) \quad \varphi(0, -v, t) = \varphi(0, rv, t), \quad v > 0$$

$$(16.13) \quad \lim_{(x,v) \rightarrow (0,0)} (\mathcal{L}\varphi)(0, 0) = \mu_* |C_*| \mathcal{A}(\varphi), \quad t \in (0, T)$$

with \mathcal{L} as in (11.3) and $\mathcal{A}(\varphi)$ defined by means of:

$$(16.14) \quad \varphi(x, v, t) = \varphi(0, 0, t) + \mathcal{A}(\varphi) F_\beta(x, v) + \psi(x, v, t), \quad \lim_{R \rightarrow 0} \frac{\sup_{x+|v|^3=R} |\psi(x, v)|}{R^\beta} = 0$$

We will obtain a well defined initial-boundary value problem with:

$$(16.15) \quad \varphi(x, v, 0) = \varphi_0(x, v)$$

16.2. The case $r > r_c$: In the case $r > r_c$ we only need to consider the case in which the boundary condition is (6.1). Due to the definition of Ω_{sup} in Section 13.4 and Proposition 15.1 we obtain that the adjoint of the problem (1.3), (1.4) with boundary condition (6.1) can be formulated as:

$$(16.16) \quad \partial_t \varphi - v \partial_x \varphi - \partial_{vv} \varphi = 0, \quad x > 0, \quad v \in \mathbb{R}$$

$$(16.17) \quad \varphi(0, -v, t) = \varphi(0, rv, t), \quad v > 0$$

$$(16.18) \quad \varphi(x, v, t) = \varphi(0, 0, t) + \psi(x, v, t), \quad \lim_{R \rightarrow 0} \sup_{x+|v|^3=R} |\psi(x, v, t)| = 0, \quad t \in (0, T)$$

complemented with:

$$(16.19) \quad \varphi(x, v, 0) = \varphi_0(x, v)$$

As a next step we will prove that the problems (16.16)-(16.19), (16.3)-(16.6), (16.7)-(16.10), (16.11)-(16.15) can be solved for suitable choices of initial data φ_0 . These solvability results will be used to define a suitable concept of solution for the boundary value problems (1.3), (1.4) with one of the boundary conditions (5.8), (5.9), (5.11), (6.1) by means of the duality formula (8.1).

17. WELL-POSEDNESS OF THE ADJOINT PROBLEMS.

In order to prove well-posedness of the problems (16.16)-(16.19), (16.3)-(16.6), (16.7)-(16.10), (16.11)-(16.15) we will use the version of Hille-Yosida Theorem that we recall in the next Section.

18. HILLE-YOSIDA THEOREM.

We will follow closely the formulation of the Hille-Yosida Theorem in [39], which is particularly well suited for the study of the adjoint problems summarized in Section 16 of Section 3.

The following results are Definition 2.1 and Proposition 2.2, Section 1 of [39].

Definition 18.1. Let Ω be a linear operator on the Banach space $C(X)$ and $\mathcal{D}(\Omega)$ its domain. We will say that Ω is a Markov pregenerator if:

- (i): $1 \in \mathcal{D}(\Omega)$, $\Omega 1 = 0$.
- (ii): $\mathcal{D}(\Omega)$ is dense in $C(X)$.
- (iii): If $f \in \mathcal{D}(\Omega)$, $\lambda \geq 0$, and $f - \lambda \Omega f = g$, then $\min_{\zeta \in X} f(\zeta) \geq \min_{\zeta \in X} g(\zeta)$.

Lemma 18.2. Suppose that the linear operator Ω on $C(X)$ satisfies that for any $f \in \mathcal{D}(\Omega)$ such that $f(\eta) = \min_{\zeta \in C(X)} f(\zeta)$, we have $\Omega f(\eta) \geq 0$. Then Ω satisfies the property (iii) in the Definition 18.1.

We recall that an operator Ω in a Banach space E is closed if its graph is closed in $E \times E$ (cf. [10]). In our specific setting this just means the following:

Definition 18.3. Ω is a closed operator if $\{(\varphi, \Omega \varphi) : \varphi \in \mathcal{D}(\Omega)\}$ is a closed set in $C(X) \times C(X)$. That is, if $\varphi_n \rightarrow \varphi^*$ and $\Omega \varphi_n \rightarrow \psi$ in $C(X)$, then $\varphi^* \in \mathcal{D}(\Omega)$ and $\psi = \Omega \varphi^*$.

The following results are just a reformulation of Proposition 2.6, Definition 2.7 and Proposition 1.3 of [39].

Lemma 18.4. *Suppose Ω is a closed Markov pregenerator. Then $\mathcal{R}(I - \lambda\Omega)$ is a closed subset of $C(X)$ for $\lambda > 0$.*

Definition 18.5. *Ω is a Markov generator if it is a closed Markov pregenerator such that $\mathcal{R}(I - \lambda\Omega) = C(X)$ for all sufficiently small positive λ .*

Definition 18.6. *$\{S(t) \mid t \geq 0\}$ is a Markov semigroup if it is a family of operators on $C(X)$ satisfying the following:*

- (i): $S(0) = I$.
- (ii): $\pi : [0, \infty) \rightarrow C(X)$ by $\pi(t) = S(t)f$ is right-continuous for every $f \in C(X)$.
- (iii): $S(t+s)f = S(t)S(s)f$ for all $f \in C(X)$, $s, t \geq 0$.
- (iv): $S(t)1 \geq 0, t \geq 0$.
- (v): $S(t)f \geq 0$ for all nonnegative $f \in C(X)$.

The Hille-Yosida Theorem provides a connection between Markov generators and Markov semigroups. The following version of this Theorem is the one in Theorem 2.9 of [39].

Theorem 18.7. *(Hille-Yosida Theorem) There is a one-to-one correspondence between Markov generators on $C(X)$ and Markov semigroups on $C(X)$. The correspondence is given by the following:*

$$\mathcal{D}(\Omega) = \left\{ f \in C(X) \mid \lim_{t \rightarrow 0^+} \frac{S(t)f - f}{t} \text{ exists} \right\}$$

and

$$\Omega f = \lim_{t \rightarrow 0^+} \frac{S(t)f - f}{t}, \quad f \in \mathcal{D}(\Omega)$$

If $f \in \mathcal{D}(\Omega)$, then $S(t)f \in \mathcal{D}(\Omega)$ and

$$\frac{d}{dt}(S(t)f) = \Omega S(t)f$$

19. COMPARISON PRINCIPLES AND TRACE PROPERTIES.

In this Section we collect several Maximum Principle properties of the operator \mathcal{L} which will be used in the following.

19.1. Basic Definitions. Due to the directionality of the transport terms in the operator \mathcal{L} we do not need to impose conditions in all the boundary of the domain where the problem is satisfied in order to obtain comparison results. In order to keep the statements simple we will formulate the Maximum Principle only for the particular class of domains in Definition 12.1.

The purpose of the following Definition is to have a suitable unified notation for the spaces which will be needed to formulate comparison principles both for the whole space X and the admissible domains Ξ .

Definition 19.1. *We will define as $L_b^\infty(X)$ the space of functions φ such that, for any compact set $K \subset [0, \infty) \times (-\infty, \infty)$ there exists a constant $C = C(K)$ such that $\|\varphi\|_{L^\infty(K)} \leq C(K)$. In the case of the domains Ξ in Definition 12.1 we will define as $L_b^\infty(\Xi)$ just the space $L^\infty(\Xi)$.*

We now define a suitable concept of sub/supersolutions in the domains Ξ and X . We recall that the class of functions $\mathcal{F}(Y)$ is as in (11.9).

Definition 19.2. Suppose that Y is, either the space X , or one of the admissible domains Ξ defined in Definition 12.1. We will say that $\varphi \in L_b^\infty(Y)$ is a supersolution for the operator $\mathcal{L}(\cdot) + \kappa$, with $\kappa \in \mathbb{R}$, if, for any $\psi \in \mathcal{F}(Y)$, $\psi \geq 0$ the following inequality holds:

$$(19.1) \quad - \int_X \varphi (\mathcal{L}^* \psi + \kappa \psi) dx dv \geq 0$$

where in the case of domains containing all or part of the line $\{x = 0\}$ we assume in addition that $\psi(0, -v) = r^2 \psi(0, rv)$, $v > 0$.

We will say that $\varphi \in L_b^\infty(Y)$ is a subsolution for the operator $\mathcal{L}\varphi + \kappa$ if $(-\varphi)$ is a supersolution for the operator $\mathcal{L}\varphi - \kappa$.

19.2. Traces and their properties. We will need to use sub and supersolutions having discontinuities in some vertical lines. This is the main reason to assume only L_b^∞ regularity in Definition 19.2. Notice that this Definition does not make any reference to boundary conditions. This is just due to the fact that it is not possible to define boundary values for a function φ which is only in L^∞ . However, it turns out to be possible to define a suitable concept of trace for the supersolutions and subsolutions in Definition 19.2. This fact will play a crucial role in the rest of the paper.

Proposition 19.3. Suppose that Y is as in Definition 19.2 and $\varphi \in L_b^\infty(Y)$ is a supersolution for the operator $\mathcal{L}(\cdot) + \kappa I$ for some $\kappa \in \mathbb{R}$. Let $\omega \in \mathcal{F}(Y)$. It is possible to modify the function φ in a set of zero measure in Y to obtain a function φ (also denoted as φ) with the following properties:

(i) Let $L_* \subset \bar{Y}$ be any horizontal segment in the sense of Definition 11.5. Then there exists $\ell_{L_*}(\varphi, \omega) \in \mathbb{R}$ such that

$$(19.2) \quad \lim_{\varepsilon \rightarrow 0^+} \left(\sup \left\{ \left| \int_L \omega \varphi ds - \ell_{L_*}(\varphi, \omega) \right| : L \subset Y \text{ horizontal segment with } \text{dist}_H(L, L_*) \leq \varepsilon \right\} \right) = 0$$

with $\text{dist}_H(L, L_*)$ as in (11.2). There exists a function $\psi_{L_*} \in L^\infty(L_*)$ such that $\ell_{L_*}(\varphi, \omega) = \int_{L_*} \psi_{L_*} \omega ds$. Moreover, (19.2) holds also for any function ω with the form $\omega(x, v) = \zeta(x) \chi_{L_*}(v)$, with $\zeta \in L^1(L_*)$ and where χ_{L_*} is the characteristic function of the set L_* .

Moreover, there exist also $m_{L_*}^+(\varphi, \omega), m_{L_*}^-(\varphi, \omega) \in \mathbb{R}$ such that, for any $v_0 \in L_*$ we have:

$$(19.3) \quad \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{L_* \times (v_0, v_0 + \varepsilon)} \omega \partial_v \varphi dx dv = m_{L_*}^+(\varphi, \omega)$$

$$(19.4) \quad \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{L_* \times (-\varepsilon + v_0, v_0)} \omega \partial_v \varphi dx dv = m_{L_*}^-(\varphi, \omega)$$

There exist Radon measures $Q_{L_*}^+, Q_{L_*}^- \in \mathcal{M}(L_*)$ such that $m_{L_*}^\pm(\varphi, \omega) = \int_{L_*} Q_{L_*}^\pm \omega ds$.

(ii) Let $L_* \subset \bar{Y}$ be any vertical segment in the sense of Definition 11.5 with $x_0 \in L_*$. Then there exists $\ell_{L_*}^+(\varphi, \omega), \ell_{L_*}^-(\varphi, \omega) \in \mathbb{R}$ such that

$$(19.5) \quad \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{L_* \times (x_0, x_0 + \varepsilon)} \omega \varphi dx dv = \ell_{L_*}^+(\varphi, \omega)$$

$$(19.6) \quad \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{L_* \times (-\varepsilon + x_0, x_0)} \omega \varphi dx dv = \ell_{L_*}^-(\varphi, \omega)$$

There exist functions $\varphi_{L_*}^+, \varphi_{L_*}^- \in L^\infty(L_*)$ such that $\ell_{L_*}^\pm(\varphi, \omega) = \int_{L_*} \varphi_{L_*}^\pm \omega ds$. Moreover (19.5), (19.6) hold for any function ω with the form $\omega(x, v) = \chi_{L_*}(x) \zeta(v)$, with $\zeta \in L^1(L_*)$ and where χ_{L_*} is the characteristic function of the set L_* .

The same results hold for subsolutions.

Proof. Suppose that L_* is a vertical line as in the statement of the Proposition and $\omega \in \mathcal{F}(Y)$. Let us denote as $\zeta = \zeta(v)$ the restriction of ω to the line L_* . Due to the continuity of ω , to prove (19.5) is equivalent to prove:

$$(19.7) \quad \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{L_* \times (x_0, x_0 + \varepsilon)} \zeta \varphi dx dv = \ell_{L_*}^+(\varphi, \omega)$$

where we assume that ζ is extended from L_* to $L_* \times (0, \varepsilon)$ assuming that $\zeta_x = 0$. Due to the smoothness of ζ we can write $\zeta = \zeta_+ - \zeta_- + R$ where ζ_+ and ζ_- are nonnegative and smooth and R is small in $L^1(L_*)$. Due to the boundedness of φ the contribution of R to the left-hand side of (19.7) can be bounded by $\|R\|_{L^1(L_*)}$. Therefore,

$$\left| \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{L_* \times (x_0, x_0 + \varepsilon)} \zeta \varphi dx dv - \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{L_* \times (x_0, x_0 + \varepsilon)} (\zeta_+ - \zeta_-) \varphi dx dv \right| \leq C \|R\|_{L^1(L_*)}$$

Then, proving formulas like (19.7) for ζ_+ and ζ_- for a functional $\ell_{L_*}^+(\varphi, \omega) = \int_{L_*} \psi_{L_*} \omega ds$ with ψ_{L_*} as in the statement of the Proposition, we would obtain (19.7) with a little remainder $C \|R\|_{L^1(L_*)}$ which can be made arbitrarily small and the result would follow for ζ . We can then restrict ourselves to the case of $\zeta \geq 0$.

We define an auxiliary test function $\bar{\omega}$ by means of:

$$\bar{\omega}(x, v) = \begin{cases} \frac{2}{\varepsilon} \left(x - \frac{\varepsilon}{2}\right) \frac{\zeta(v)}{v} & \text{if } \frac{\varepsilon}{2} \leq x - x_0 \leq \varepsilon \\ \frac{1}{\varepsilon} (2\varepsilon - x) \frac{\zeta(v)}{v} & \text{if } \varepsilon \leq x - x_0 \leq 2\varepsilon \\ 0 & \text{otherwise} \end{cases}$$

A density argument shows that (19.1) holds for the test function $\bar{\omega}$ in spite of the fact that this function is not in $\mathcal{F}(Y)$. Therefore, using 19.1, we obtain that

$$\frac{2}{\varepsilon} \int_{L_* \times (x_0 + \frac{\varepsilon}{2}, x_0 + \varepsilon)} \varphi \zeta - \frac{1}{\varepsilon} \int_{(x_0 + \varepsilon, x_0 + 2\varepsilon)} \varphi \zeta \geq -C\varepsilon$$

where C is a constant depending of $\|\varphi\|_{L^\infty}$, $\|\zeta_{vv}\|_{L^\infty}$, $\text{supp}(\zeta)$, L_* and κ . It then follows that for any $\varepsilon_0 > 0$ small the sequence $\left\{ \frac{2^n}{\varepsilon_0} \int_{L_* \times (x_0 + 2^{-n}\varepsilon_0, x_0 + 2^{-n+1}\varepsilon_0)} \varphi \zeta + C2^{-n+1}\varepsilon_0 \right\}$ is an increasing bounded sequence. Therefore, the limit

$$\lim_{n \rightarrow \infty} \frac{2^n}{\varepsilon_0} \int_{L_* \times (x_0 + 2^{-n}\varepsilon_0, x_0 + 2^{-n+1}\varepsilon_0)} \varphi \zeta = \ell_0$$

exists. This implies the existence of the limit $\lim_{n \rightarrow \infty} \frac{2^n}{\varepsilon_0} \int_{L_* \times (x_0, x_0 + 2^{-n}\varepsilon_0)} \varphi \zeta$, since we can write:

$$\frac{2^n}{\varepsilon_0} \int_{L_* \times (x_0, x_0 + 2^{-n}\varepsilon_0)} \varphi \zeta = \frac{2^n}{\varepsilon_0} \sum_{k=0}^{\infty} \int_{L_* \times (x_0 + 2^{-n-k-1}\varepsilon_0, x_0 + 2^{-n-k}\varepsilon_0)} \varphi \zeta$$

Using then the approximation

$$\int_{L_* \times (x_0 + 2^{-n-k-1}\varepsilon_0, x_0 + 2^{-n-k}\varepsilon_0)} \varphi \zeta = 2^{-n-k-1}\varepsilon_0 (\ell_0 + o(1)) \text{ as } n \rightarrow \infty$$

we obtain $\lim_{n \rightarrow \infty} \left(\frac{2^n}{\varepsilon_0} \int_{L_* \times (x_0, x_0 + 2^{-n}\varepsilon_0)} \varphi \zeta \right) = \ell_0$. It only remains to show that the limit is independent of ε_0 for any $\varepsilon_0 \in (0, 1)$. Let us take $\varepsilon_1 \neq \varepsilon_0$. Let ℓ_1 be $\lim_{n \rightarrow \infty} \frac{2^n}{\varepsilon_1} \int_{L_* \times (x_0, x_0 + 2^{-n}\varepsilon_1)} \varphi \zeta = \ell_1$. Suppose that n is any large integer. For any such n , we select another integer m such

that $\varepsilon_0 2^{-m} < \varepsilon_1 2^{-n-1}$. We consider a new test function $\bar{\omega}$ given by:

$$\bar{\omega}(x, v) = \begin{cases} \frac{2^n}{\varepsilon_0} x \frac{\zeta(v)}{v} & \text{if } 0 \leq x - x_0 \leq \varepsilon_0 2^{-n} \\ \frac{\zeta(v)}{v} & \text{if } \varepsilon_0 2^{-m} \leq x - x_0 \leq \varepsilon_1 2^{-n-1} \\ \frac{2^{n+1}}{\varepsilon_1} (\varepsilon_1 2^{-n} - x) \frac{\zeta(v)}{v} & \text{if } \varepsilon_1 2^{-n-1} \leq x - x_0 \leq \varepsilon_1 2^{-n} \\ 0 & \text{otherwise} \end{cases}$$

Using (19.1) we obtain:

$$\frac{2^m}{\varepsilon_0} \int_{L_* \times (x_0, x_0 + \varepsilon_0 2^{-m})} \varphi \zeta - \frac{2^{n+1}}{\varepsilon_1} \int_{L_* \times (x_0 + \varepsilon_1 2^{-n-1}, x_0 + \varepsilon_1 2^{-n})} \varphi \zeta \geq -C \varepsilon_1 2^{-n}$$

for some constant C independent of n, m . Taking the limit $n \rightarrow \infty$ we obtain $\ell_0 \geq \ell_1$. Reversing the role of $\varepsilon_0, \varepsilon_1$ we would obtain $\ell_1 \geq \ell_0$ and the independence of the limit on the choice of ε_0 follows.

Notice that the functional which assigns the function ζ to the limit

$$\lim_{n \rightarrow \infty} \frac{2^n}{\varepsilon_0} \int_{L_* \times (x_0, x_0 + 2^{-n} \varepsilon_0)} \varphi \zeta$$

can be extended to linear functional in $L^1(\rho, R)$ for any $0 < \rho < R < \infty$. Since the dual of $L^1(\rho, R)$ is $L^\infty(\rho, R)$ we obtain the representation formula $\ell_{L_*}^+(\varphi, \omega) = \int_{L_*} \varphi_{L_*}^+ \omega ds$. The proof of (19.6) is similar.

Suppose now that $L_* \subset \bar{Y}$ is a horizontal line and $\omega \in \mathcal{F}(Y)$. Suppose that $v = v_0$ in L_* . We denote as $\zeta = \zeta(x)$ the restriction of ω to the line L_* . The continuity of ω implies that (19.2) is equivalent to:

$$\lim_{\varepsilon \rightarrow 0^+} \int_L \zeta \varphi ds = \ell_{L_*}(\varphi, \omega)$$

where $\text{dist}_H(L, L_*) = \varepsilon$. We can assume, arguing as in the case of vertical lines, that $\zeta \geq 0$. We label the lines L by means of the value of v on them, i.e $L = L(v)$. We can define a function $v \rightarrow \Phi(v) = \int_{L(v)} \zeta \varphi ds$. We then consider a test function $\omega(x, v) = \zeta(x) \beta(v)$ for some smooth $\beta \geq 0$. Inequality (19.1) with $\psi = \omega$ yields:

$$\int \Phi(v) \beta_{vv} dv \leq C \int \beta dv$$

where the constant C depends on ζ and its derivatives and φ . This implies that the function $\Phi - Cv^2$ is concave and the limit $\lim_{v \rightarrow v_0} \Phi(v)$ exists. This limit defines a linear functional in the set of functions ζ which can be continuously extended to $L^1(L_*)$ due to the boundedness of φ . Therefore $\ell_{L_*}(\varphi, \omega) = \int_{L_*} \varphi_{L_*}^+ \omega ds$ and (19.2) follows.

Moreover, the concavity of $\Phi - Cv^2$ implies that $\Phi'(v)$ exists except at a countable set. Then $\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{v_0}^{v_0 + \varepsilon} \Phi'(v) dv$ exists and we have:

$$\Phi'(v) = \beta_v(v) \int_{L(v)} \zeta(x) \varphi ds + \beta(v) \int_{L(v)} \zeta(x) \varphi_v ds \quad \text{a.e. } v \in (v_0 - \varepsilon, v_0 + \varepsilon)$$

where we use the fact that $\int_{L(v)} \zeta(x) \varphi(x, v) ds = \int_I \zeta(x) \beta(v) \varphi(x, v) ds$ for a fixed interval I for each v . Using (19.2) we obtain the existence of the limit on the left-hand side of (19.3). This gives (19.3). The proof of (19.4) is similar. The existence of the measures $Q_{L_*}^\pm$ is a consequence of the Riesz representation Theorem.

The proof of the results for subsolutions can be obtained in a similar way with minor changes. \square

Remark 19.4. We will denote the function φ_{L_*} in the case (i) as the trace of φ in the horizontal line L_* . We will denote the functions $\varphi_{L_*}^+, \varphi_{L_*}^-$ in the case (ii) as the right-side trace and left-side traces of φ respectively. Given a domain Ξ we will use also the terminology "interior traces" to denote the trace from the domain Ξ . We will denote the functions $Q_{L_*}^+, Q_{L_*}^-$ as the right-side trace and left-side traces of $\partial_v \varphi$ respectively.

We remark that Proposition 19.3 allows us to obtain a suitable definition of boundary data for arbitrary supersolutions of $\mathcal{L}(\cdot) + \kappa$ in the sense of Definition 19.2 defined in an admissible domain Ξ .

Definition 19.5. Suppose that Ξ is one admissible domain in the sense of Definition 12.1 and φ is a supersolution of $\mathcal{L}(\cdot) + \kappa$ in the sense of Definition 19.2. The admissible boundary $\partial_a \Xi$ consists in the union of some horizontal and vertical segments contained in \bar{Y} . The function $\bar{\varphi} \in L^\infty(\partial_a \Xi)$ obtained patching the horizontal and vertical traces will be termed the boundary value of φ in $\partial_a \Xi$.

We will use the following notation. Given any open set $\mathcal{Z} \subset \{(x, v) : x > 0, v \in \mathbb{R}\}$

$$(19.8) \quad \begin{aligned} \mathcal{Z}^+ &= \{(x, v) \in \mathcal{Z} : v > -x\} \\ \mathcal{Z}^- &= \{(x, v) \in \mathcal{Z} : v < x\} \end{aligned}$$

We will denote as \mathcal{Z}^\pm one of the sets \mathcal{Z}^+ or \mathcal{Z}^- . Given that one of the main ideas used later is an adaptation of the classical Perron's method for harmonic functions, we need to prove that the maximum of subsolutions is a subsolution. Since the subsolutions considered here might have discontinuities, this requires to understand the traces of functions that are obtained by means of maxima of subsolutions. This will be achieved in the following Lemmas.

The following technical Lemma yields a general property of L^∞ functions with traces. They are not required to be sub or supersolutions.

Lemma 19.6. Suppose that φ_1, φ_2 are respectively two functions in L^∞ defined respectively in the open sets $\mathcal{W}_1^\pm, \mathcal{W}_2^\pm$ with $\mathcal{W}_1^\pm \cap \mathcal{W}_2^\pm \cap \{x = 0\} \neq \emptyset$. Let us assume that the boundary values of $\varphi_1, \varphi_2, \max\{\varphi_1, \varphi_2\}$ at $\mathcal{W}_1^\pm \cap \mathcal{W}_2^\pm \cap \{x = 0\}$ can be defined in the sense of traces, i.e. there exist functions $\varphi_1^\pm, \varphi_2^\pm, (\max\{\varphi_1, \varphi_2\})^\pm \in L^\infty(\mathcal{W}_1^\pm \cap \mathcal{W}_2^\pm \cap \{x = 0\})$ such that

$$(19.9) \quad \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{[\mathcal{W}_1^\pm \cap \mathcal{W}_2^\pm \cap \{x=0\}] \times (0, \varepsilon)} \varphi_k \zeta = \int_{\mathcal{W}_1^\pm \cap \mathcal{W}_2^\pm \cap \{x=0\}} \varphi_k^\pm \zeta, \quad k = 1, 2$$

$$(19.10) \quad \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{[\mathcal{W}_1^\pm \cap \mathcal{W}_2^\pm \cap \{x=0\}] \times (0, \varepsilon)} \max\{\varphi_1, \varphi_2\} \zeta = \int_{\mathcal{W}_1^\pm \cap \mathcal{W}_2^\pm \cap \{x=0\}} (\max\{\varphi_1, \varphi_2\})^\pm \zeta$$

for any function $\zeta \in L^1_{loc}(\mathcal{W}_1^\pm \cap \mathcal{W}_2^\pm \cap \{x = 0\})$. Then:

$$(19.11) \quad (\max\{\varphi_1, \varphi_2\})^\pm \geq \max\{\varphi_1^\pm, \varphi_2^\pm\}$$

Remark 19.7. Due to the nonlinearity of the operator $\max(\cdot)$, we cannot hope to replace the sign \geq by $=$ in (19.11), because the convergence in (19.9) is only a weak convergence. Indeed, if we define the functions $\varphi_1(x, v) = \text{sgn}(\cos(2^n v))$ for $2^{-n-1} \leq x < 2^{-n}$ and $\varphi_2 = 0$ and we denote as $(\cdot)^+$ the traces at $x = 0$, we have $\varphi_1^+ = \varphi_2^+ = 0$, but $(\max\{\varphi_1, \varphi_2\})^+ = \frac{1}{2}$.

Remark 19.8. Proposition 19.3 implies (19.9) for sub and supersolutions for any function $\zeta \in L^1((\pm\rho, \pm R) \cap \mathcal{W}_1^\pm \cap \mathcal{W}_2^\pm)$ and $k = 1, 2$ and $0 < \rho < R < \infty$ arbitrary.

Proof. Our goal is to show that $\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{[\mathcal{W}_1^\pm \cap \mathcal{W}_2^\pm \cap \{x=0\}] \times (0, \varepsilon)} \max\{\varphi_1, \varphi_2\} \zeta \geq \int_{\mathcal{W}_1^\pm \cap \mathcal{W}_2^\pm \cap \{x=0\}} \max\{\varphi_1^\pm, \varphi_2^\pm\} \zeta$ for any test function $\zeta \geq 0$. In order to simplify the argument we define $F = \varphi_1 - \varphi_2$. Then

(19.9) becomes:

$$(19.12) \quad \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{[\mathcal{W}_1^\pm \cap \mathcal{W}_2^\pm \cap \{x=0\}] \times (0, \varepsilon)} F \zeta = \int_{\mathcal{W}_1^\pm \cap \mathcal{W}_2^\pm \cap \{x=0\}} F^\pm \zeta$$

for any $\zeta \in L^1((\pm\rho, \pm R) \cap \mathcal{W}_1^\pm \cap \mathcal{W}_2^\pm \cap \{x=0\})$, $0 < \rho < R < \infty$ with $F^\pm \in L^\infty(v \gtrless 0)$. We then need to prove:

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{[\mathcal{W}_1^\pm \cap \mathcal{W}_2^\pm \cap \{x=0\}] \times (0, \varepsilon)} \max\{F, 0\} \zeta \geq \int_{\mathcal{W}_1^\pm \cap \mathcal{W}_2^\pm \cap \{x=0\}} \max\{F^\pm, 0\} \zeta$$

or equivalently $\max\{F^\pm, 0\} \leq (\max\{F, 0\})^\pm$. To this end we argue as follows. We will restrict all the analysis to arbitrary compact sets of $\{v \gtrless 0\}$, say $[\pm\rho, \pm R]$, with $0 < \rho < R < \infty$, although this restriction will not be made explicit in the formulas for simplicity. Then, using (19.12):

$$\begin{aligned} \int_{\mathcal{W}_1^\pm \cap \mathcal{W}_2^\pm \cap \{x=0\}} F^\pm \zeta &= \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{[\mathcal{W}_1^\pm \cap \mathcal{W}_2^\pm \cap \{x=0\}] \times (0, \varepsilon)} F \zeta \\ &\leq \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{[\mathcal{W}_1^\pm \cap \mathcal{W}_2^\pm \cap \{x=0\}] \times (0, \varepsilon)} \max\{F, 0\} \zeta \\ &= \int_{\mathcal{W}_1^\pm \cap \mathcal{W}_2^\pm \cap \{x=0\}} (\max\{F, 0\})^\pm \zeta \end{aligned}$$

and similarly:

$$0 \leq \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{[\mathcal{W}_1^\pm \cap \mathcal{W}_2^\pm \cap \{x=0\}] \times (0, \varepsilon)} \max\{F, 0\} \zeta = \int_{\mathcal{W}_1^\pm \cap \mathcal{W}_2^\pm \cap \{x=0\}} (\max\{F, 0\})^\pm \zeta$$

for any test function $\zeta \geq 0$, whence $(\max\{F, 0\})^\pm \geq \max\{F^\pm, 0\}$. Then

$$(\max\{\varphi_1 - \varphi_2, 0\})^\pm \geq \max\{\varphi_1^\pm - \varphi_2^\pm, 0\}.$$

Therefore

$$\begin{aligned} (\max\{\varphi_1, \varphi_2\})^\pm &= (\varphi_2 + \max\{\varphi_1 - \varphi_2, 0\})^\pm \\ &= \varphi_2^\pm + (\max\{\varphi_1 - \varphi_2, 0\})^\pm \\ &\geq \varphi_2^\pm + \max\{\varphi_1^\pm - \varphi_2^\pm, 0\} = \max\{\varphi_1^\pm, \varphi_2^\pm\} \end{aligned}$$

whence the result follows. \square

In order to prove that the maximum of subsolutions is a subsolution we need to be able to replace the inequality in (19.11) by an equality sign in the case of the trace operator $(\cdot)^-$ defined in the domains \mathcal{W}^\pm . It turns out that this is possible for the subsolutions defined in Definition 23.8. The key point is the fact that it is possible to derive some regularity estimates for subsolutions of (23.11), (23.12). More precisely, the following Lemma shows that for domains contained in $\{v < 0\}$ the traces in some vertical lines can be defined not just in the weak topology but in L^1 .

Lemma 19.9. *Suppose that Ξ is an admissible domain as in Definition 12.1 with $\Xi \subset \{v < 0\}$. Suppose that $\varphi \in L^\infty(\Xi)$ satisfies in the sense of distributions:*

$$(19.13) \quad \mathcal{L}\varphi = \nu - F$$

where $\nu \geq 0$ is a Radon measure in Ξ and $F \in L^\infty(\Xi)$. We assume that $\int_\Xi \nu < \infty$. Suppose that the admissible boundary of Ξ is:

$$\partial_a \Xi = (\{x_1\} \times [-\beta, -\alpha]) \cup (\{-\beta\} \times [x_1, x_2]) \cup (\{-\alpha\} \times [x_1, x_2])$$

for some $0 < \alpha < \beta$, $0 \leq x_1 < x_2$. Then, it is possible to define $\varphi(x_1, \cdot)$ in the sense of trace as in Proposition 19.3 and we have:

$$(19.14) \quad \varphi(x, \cdot) \rightarrow \varphi(x_1, \cdot) \text{ as } x \rightarrow (x_1)^+ \text{ in } L^1(-\beta, -\alpha)$$

Proof. Equation (19.13) is a forward parabolic equation for increasing x , due to the fact that v is negative. Moreover, the variable $\Phi = v\varphi$ satisfies a parabolic equation with the form of conservation law, more precisely we have:

$$(19.15) \quad \partial_x \Phi + \partial_v^2 \left(\frac{\Phi}{v} \right) = -F + \nu$$

where F is bounded and $\int_{(x_1, x_1 + \delta_0) \times K} \nu(dx dv) \leq C_0$. Let us denote as $G(v, w, x, y)$ the fundamental solution associated to the left-hand side of (19.15) in the domain $(x_1, x_1 + \delta_0) \times K$, where $K = [-\beta, -\alpha]$. Notice that the function Φ is well defined in $(x_1, x_1 + \delta_0) \times \partial K$ and in $\{x_1\} \times K$ in the sense of traces due to Proposition 19.3. We can derive a representation formula for Φ using the fundamental solution G (cf. [18]):

$$(19.16) \quad \Phi(x, v) = \int_{\partial_a((x_1, x) \times K)} G(v, w, x, y) \Phi(y, w) + \int_{(x_1, x) \times K} G(v, w, x, y) [-F + \nu]$$

Due to the boundedness of Φ , it is enough to prove that $\varphi(x, \cdot) \rightarrow \varphi(x_1, \cdot)$ as $x \rightarrow (x_1)^+$ in $L^1(\bar{K})$ for any $\bar{K} \subset\subset K$. The convergence of the first term on the right-hand side of (19.16) follows from classical results for parabolic equations, as well as the convergence of $\int_{(x_1, x) \times K} G(v, w, x, y) F$ to zero as $x \rightarrow (x_1)^+$. It only remains to prove the convergence of $\int_{(x_1, x_1 + \delta_0) \times K} G(v, w, x, y) \nu(w, y)$ to zero as $x \rightarrow (x_1)^+$. The contribution to the trace of this term is given by the limit of:

$$(19.17) \quad \frac{1}{\varepsilon} \int_{(x_1, x_1 + \varepsilon]} dx \int_{(x_1, x]} dy \int_K dw \nu(w, y) \int_{\bar{K}} dv G(v, w, x, y)$$

as $\varepsilon \rightarrow 0^+$. Using Fubini's Theorem it is possible to rewrite this integral as:

$$(19.18) \quad \int_{(x_1, x_1 + \varepsilon]} dy \int_K dw \nu(w, y) \frac{1}{\varepsilon} \int_{(y, x_1 + \varepsilon]} dx \int_{\bar{K}} dv G(v, w, x, y)$$

Standard regularity properties for Radon measures yield

$$(19.19) \quad \int_{(x_1, x_1 + \varepsilon]} dy \int_{\bar{K}} dw \nu(w, y) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0^+$$

whence the integral in (19.18) converges to zero as $\varepsilon \rightarrow 0$ due to the boundedness of $\int_{\bar{K}} dv G(v, w, x, y)$. Moreover, using the representation formula (19.16) with K replaced by \bar{K} as well as the fact that $\int G(v, w, x, y) dv$ is uniformly bounded for $x, y \in (x_1, x_1 + \varepsilon)$ and $w \in \bar{K}$ we then obtain, taking into account also (19.19) that

$$\int_{\bar{K}} dv \int_{(x_1, x) \times \bar{K}} G(v, w, x, y) F \rightarrow 0$$

and

$$\int_{\bar{K}} dv \int_{(x_1, x) \times \bar{K}} G(v, w, x, y) \nu(w, y) \rightarrow 0 \text{ as } x \rightarrow x_1^+.$$

Then the result follows. \square

Lemma 19.10. Suppose that φ_1, φ_2 with $\varphi_k \in L^\infty(\mathcal{W}_k^-)$, $k = 1, 2$ are two subsolutions of (23.11), (23.12) in the sense of Definition 23.8, where the domains $\mathcal{W}_1, \mathcal{W}_2$ satisfy $\mathcal{W}_1^- \cap$

$\mathcal{W}_2^- \cap \{x = 0\} \neq \emptyset$. Then the boundary values of φ_1, φ_2 defined in $\mathcal{W}_1^- \cap \mathcal{W}_2^- \cap \{x = 0\}$ in the sense of traces by means of Proposition 19.3 satisfy:

$$(19.20) \quad (\max \{\varphi_1, \varphi_2\})^+ = \max \{\varphi_1^+, \varphi_2^+\} \quad \text{a.e. in } \mathcal{W}_1^- \cap \mathcal{W}_2^- \cap \{x = 0\}$$

Proof. The inequality (23.16) implies that there exists a measure $\nu \in \mathcal{M}_+(\mathcal{W}_1 \cap \mathcal{W}_2)$ such that:

$$(19.21) \quad \varphi - \lambda \mathcal{L}(\varphi) - g = -\nu$$

Due to Lemma 19.9 we obtain $\varphi_k \rightarrow (\varphi_k)^+$ as $x \rightarrow 0^+$ in $L^1(K)$ for $K \subset \mathcal{W}_1^- \cap \mathcal{W}_2^- \cap \{x = 0\}$, $k = 1, 2$. Then $\max \{\varphi_1, \varphi_2\} \rightarrow \max \{\varphi_1^+, \varphi_2^+\}$ in $L^1(K)$ due to Lebesgue dominated convergence Theorem. Using the definition of Traces (see Remark 19.4) it then follows that $(\max \{\varphi_1, \varphi_2\})^+$ is well defined and (19.20) holds. \square

We will use also the following continuity result for the traces.

Lemma 19.11. *Suppose that we have a sequence of bounded functions $\{\varphi_n\}$ satisfying $-\mathcal{L}\varphi_n = \mu_n + g_n$ where $g_n \in C(\Xi)$, the functions g_n are uniformly bounded and $\mu_n \geq 0$ are Radon measures. We assume that Ξ is an admissible domain. Suppose also that the sequence $\{\varphi_n\}$ converges to φ in the weak topology and $\mu_n \rightarrow 0$ also in the weak topology. Suppose that Γ is a curve made of horizontal and vertical lines contained in Ξ . Then, given any test function $\psi \in C^\infty(\Xi)$, the following convergence property holds for the traces of φ_n and its derivatives at the curve Γ*

$$\int_{\Gamma} (n_v \psi \partial_v \varphi_n - n_v \partial_v \psi \varphi_n + v \varphi_n \psi n_x) ds \rightarrow \int_{\Gamma} (n_v \psi \partial_v \varphi - n_v \partial_v \psi \varphi + v \varphi \psi n_x) ds$$

as $n \rightarrow \infty$.

Remark 19.12. *The traces of the functions φ_n can be defined at any side of the boundary Γ .*

Proof. We can first subtract from the functions φ_n the solutions of the equations $-\mathcal{L}\varphi_n = g_n$ with zero boundary conditions in the admissible boundary of Ξ . These solutions are uniformly bounded and smooth at the curve Γ and then, the corresponding traces have the desired convergence properties. We can then assume without loss of generality that $-\mathcal{L}\varphi_n = \mu_n$.

Given any other curve $\tilde{\Gamma}$ with dimensions comparable to Γ such that $\Gamma \cup \tilde{\Gamma} = \partial\Xi$ for some admissible domain Ξ we can estimate the difference between the fluxes on $\tilde{\Gamma}$ and Γ integrating by parts and using the equation $-\mathcal{L}\varphi_n = \mu_n$. The contribution due to μ_n tends to zero, due to the weak convergence of the measures. Some of the integrals $\int_{\Gamma} [\cdot \cdot] ds$ contain integrations of the functions φ_n . The integrals of $\varphi_n v$ and related functions can be represented by means of the integrals in horizontal and vertical lines using Fubini. In particular, it is possible to approximate the integral $\int_{\Gamma} [\cdot \cdot] ds$ by similar integrals in other contours $\tilde{\Gamma}$ due to the weak convergence of the functions φ_n as well as the fact that the difference of values between these integrals converges to zero as $n \rightarrow \infty$, because these differences are proportional to the integrals of μ_n on two-dimensional domains which converge to zero by assumption as $n \rightarrow \infty$. In order to conclude the argument we need to obtain suitable convergences for the integrals containing the derivatives $\partial_v \varphi_n$. This regularity follows from the hypoellipticity properties in Theorem 12.4. The same argument above, which allows to transform convergences in two-dimensional domains into convergences for $\int_{\Gamma} [\cdot \cdot] ds$ using the fact that the difference between these integrals converges to zero, gives the result. \square

20. WELL-POSEDNESS OF THE DIRICHLET PROBLEM IN ADMISSIBLE DOMAINS AND SOME REPRESENTATION FORMULAS.

We now prove the solvability of the Dirichlet problem:

$$(20.1) \quad (\varphi - \lambda \mathcal{L}\varphi) = g \text{ in } \Xi, \varphi \in C(\bar{\Xi}), \varphi(0, rv) = \varphi(0, -v), v > 0, (0, v) \in \bar{\Xi}$$

$$(20.2) \quad \varphi = h \text{ on } \partial_a \Xi$$

where $\lambda > 0$, Ξ is any of the admissible domains given in Definition 12.1, $\partial_a \Xi$ is the corresponding admissible boundary and $h \in L^\infty(\partial_a \Xi)$. Notice that, if $\bar{\Xi} \cap \{x = 0\} = \emptyset$ the condition on the boundary values $(0, v) \in \bar{\Xi}$ is empty. We will derive a representation formula for the solutions of (20.1), (20.2) by means of the stochastic process associated to the operator \mathcal{L} . To this end, we first introduce some notation for the solutions of some stochastic differential equations. We will write in the following $\xi = (X, V)$ for brevity.

Definition 20.1. *We will denote as $\xi_t = (X, V)$ the unique strong solution of the Stochastic Differential Equation:*

$$(20.3) \quad d\xi = d \begin{pmatrix} X \\ V \end{pmatrix} = \begin{pmatrix} -V \\ 0 \end{pmatrix} dt + \sqrt{2} \begin{pmatrix} 0 \\ dB_t \end{pmatrix}$$

where B_t is the Brownian motion. We will denote as $\mathbb{P}^{(x,v)}$, $\mathbb{E}^{(x,v)}$ the probability and the mean value associated to the solution of (20.3) with initial value $\xi(0) = (X, V)(0) = (x, v)$.

The existence of the stochastic process ξ in Definition 20.1 is a standard consequence of the theory of Stochastic Differential Equations (cf. [46]). Given an admissible domain Ξ as in Definition 12.1, we will denote as $\tau(\Xi)$ the stopping time associated to the process ξ in Definition 20.1 defined as:

$$(20.4) \quad \tau(\Xi) = \inf \{t \geq 0 : \xi(t) \notin \Xi\}$$

We will prove the following result:

Proposition 20.2. *Suppose that $g \in C(\bar{\Xi})$, Ξ is one of the admissible domains in Definition 12.1 and $h \in L^\infty(\partial_a \Xi)$ where $\partial_a \Xi$ is the corresponding admissible boundary of Ξ . Then, there exists a unique solution of the problem (20.1), (20.2) where the boundary condition (20.2) is achieved in the sense of trace defined in Remark 19.4. Moreover, the solution is given by the following representation formula:*

$$(20.5) \quad \varphi(x, v) = \mathbb{E}^{(x,v)} \left(e^{-\frac{\tau(\Xi)}{\lambda}} h \left(\xi_{\tau(\Xi)}^{(x,v)} \right) \right) + \frac{1}{\lambda} \mathbb{E}^{(x,v)} \left(\int_0^{\tau(\Xi)} e^{-\frac{t}{\lambda}} g \left(\xi_t^{(x,v)} \right) dt \right),$$

where $\tau(\Xi)$ is as in (20.4).

Proof. The representation formula (20.5) is standard in the Theory of Diffusion Processes. It might be found, for instance in [53], Theorem 5.1. We assume that h has been extended as zero at the set $\partial \Xi \setminus \partial_a \Xi$. Notice that the stochastic differential equation (20.3) implies that $\mathbb{P}^{(x,v)} \left(\left\{ \xi_{\tau(\Xi)}^{(x,v)} \in \partial \Xi \setminus \partial_a \Xi \right\} \right) = 0$. Indeed, this follows from the continuity of the paths associated to the process ξ_t as well as the fact that (20.3) implies that with probability one $X(\cdot)$ is differentiable and $X'(t) < 0$ if $V(t) > 0$ and $X'(t) > 0$ if $V(t) < 0$. Notice that in order to obtain (20.2) in the sense of traces it is enough to prove that the boundary values are taken in the L^1 sense. On the other hand, the formula (20.5) yields a solution of the boundary value problem (20.1), (20.2) for $h \in C(\partial_a \Xi)$. Therefore, the fact that (20.5) yields the solution of (20.1), (20.2) in the sense of traces follows from the density of the continuous functions in $L^\infty(\partial_a \Xi)$ using the norm $L^1(\partial_a \Xi)$.

Uniqueness of solutions follows from the maximum principle. Indeed, the difference of two solutions of (20.1), (20.2) satisfies $\varphi - \lambda \mathcal{L}\varphi = 0$ in Ξ , $\varphi = 0$ on $\partial_a \Xi$. Using classical regularity theory for parabolic equations we can prove that $\varphi \in C^\infty$ at all the points of the boundary of Ξ , except at the singular points (i.e. the points of the boundary where $v = 0$, namely $\partial \Xi \cap \{v = 0\}$) and at the points of the boundary of $\partial \Xi$ where the boundary is not C^∞ (i.e. the corner-like points of $\partial \Xi$). Notice that $\psi = e^{\frac{t}{\lambda}} \varphi$ is a bounded solution of $\psi_t = \mathcal{L}\psi$ satisfying $\psi = 0$ on $\partial_a \Xi$. We can then argue exactly as in the proof of Lemma 3.13 of [31] to show that for any $\varepsilon > 0$ we have $|\psi| \leq \varepsilon$ in a small neighbourhood of the singular points $\partial \Xi \cap \{v = 0\}$ and $0 \leq t \leq 1$. Similarly, classical regularity theory for parabolic equations yields also the inequality $|\psi| \leq \varepsilon$ in a neighbourhood small enough of the corner-like points of $\partial \Xi$. Therefore $|\varphi| \leq \varepsilon$ in a neighbourhood of the singular point and the corner like points of $\partial \Xi$. Using the regularity properties of the function φ it turns out that we apply the classical argument yielding the maximum principle (including a regularizing factor $\pm \delta v^2 \pm \delta x$ with $\delta > 0$ small if the derivatives do not satisfy strict inequalities at the points where they reach maximum or minimum values. It then follows that $|\varphi|$ is bounded by its values in $\partial_a \Xi$ or at the neighbourhoods indicated above of the singular point or the corner-like points. Since we have $\varphi = 0$ on $\partial_a \Xi$ it then follows that $|\varphi| \leq \varepsilon$ in Ξ . Since ε is arbitrarily small it then follows that $\varphi = 0$ and the uniqueness of (20.1), (20.2) follows. \square

Remark 20.3. *In the case of admissible domains contained in one of the half-planes $\{v > 0\}$ or $\{v < 0\}$ we can prove existence and uniqueness of solutions using the classical theory of parabolic equations.*

We will use a similar representation formula to the one obtained in Proposition 20.2 for the adjoint problem $\psi - \lambda \mathcal{L}^*(\psi) = g$. Notice that in the case of the adjoint problem the admissible boundary of the domains Ξ is not $\partial_a^* \Xi$ but the adjoint admissible boundary $\partial_a^* \Xi$ defined in Definition 12.1.

Lemma 20.4. *Suppose*

$$(20.6) \quad \varphi - \lambda \mathcal{L}(\varphi) = \nu \quad \text{in } \Xi, \quad \varphi = 0 \quad \text{on } \partial_a \Xi,$$

where $\lambda > 0$, Ξ is any of the admissible domains given in Definition 12.1, $\partial_a \Xi$ is the corresponding admissible boundary and $\nu \geq 0$ is a Radon measure. and satisfies $\int_S \nu > 0$ for some compact set $S \subset \Xi$, with $\text{dist}(S, \partial \Xi) \geq \frac{d(\Xi)}{4}$, where $d(\Xi)$ is defined as

$$(20.7) \quad d(\Xi) = \min \{ \text{dist}(L_1, L_2) : L_1, L_2 \text{ are parallel lines and } \Xi \text{ is contained between them} \}$$

Then there exists a constant $C_* > 0$, depending only on $\text{diam}(\Xi)$, $d(\Xi)$, and $\max_{(x,v) \in \Xi} |v|$ such that:

$$\varphi \geq C_* \int_S \nu > 0.$$

in the set $D = \left\{ (x, v) \in \Xi : \text{dist}((x, v), \partial_a \Xi) \geq \frac{d(\Xi)}{4} \right\}$.

Remark 20.5. *Notice that we assume $\text{dist}(S, \partial \Xi) \geq \frac{d(\Xi)}{4}$ where $\partial \Xi$ is the whole topological boundary of Ξ and we obtain estimates for φ in the set D , which is separated from the admissible boundary of Ξ at least the distance $\frac{d(\Xi)}{4}$.*

Proof. We prove the result by duality. To this end we define a test function ψ by means of the solution of the problem:

$$(20.8) \quad \psi - \lambda \mathcal{L}^*(\psi) = \zeta \quad \text{in } \Xi, \quad \psi = 0 \quad \text{on } \partial_a^* \Xi$$

where $\zeta \in C^\infty$, $\zeta = 0$ if $(x, v) \notin \Xi \setminus D$, $0 \leq \zeta \leq 1$, $\zeta = 1$ in an admissible domain $\hat{D} \subset \Xi$ with $\text{dist}(\hat{D}, \partial_a \Xi) = \frac{3d(\Xi)}{8}$ and $\hat{D} \subset\subset D$. We can obtain a representation formula for ψ analogous to (20.5) in which we just replace the stochastic process $\xi_t = (X, V)$ by one solving the Stochastic Differential Equation:

$$d\xi = d \begin{pmatrix} X \\ V \end{pmatrix} = \begin{pmatrix} V \\ 0 \end{pmatrix} dt + \sqrt{2} \begin{pmatrix} 0 \\ dB_t \end{pmatrix}$$

Notice that in this case we have $\mathbb{P}^{(x,v)} \left(\left\{ \xi_{\tau(\Xi)}^{(x,v)} \in \partial \Xi \setminus \partial_a^* \Xi \right\} \right) = 0$. Therefore, since $\psi = 0$ at $\partial_a^* \Xi$ we have:

$$(20.9) \quad \psi(x, v) = \frac{1}{\lambda} \mathbb{E}^{(x,v)} \left(\int_0^{\tau(\Xi)} e^{-\frac{t}{\lambda}} \zeta \left(\xi_t^{(x,v)} \right) dt \right)$$

On the other hand, if we denote by χ_S the characteristic function of S and we define $\bar{\varphi}$ as the unique solution of:

$$(20.10) \quad \bar{\varphi} - \lambda \mathcal{L}(\bar{\varphi}) = \nu \chi_S \text{ in } \Xi, \quad \bar{\varphi} = 0 \text{ on } \partial_a \Xi$$

Existence of solutions of this problem can be obtained approximating the nonnegative measure ν by a sequence of C^∞ functions ν_n such that $\nu_n \rightarrow \nu$ as $n \rightarrow \infty$. We denote the corresponding solutions of (20.10) with source term $\nu_n \chi_S$ as $\bar{\varphi}_n$. We have $\bar{\varphi}_n \geq 0$, $\bar{\varphi}_n$ is a C^∞ function at the interior of the domain $\Xi \setminus \bar{S}$. Integrating the corresponding equations (20.10) in Ξ we obtain:

$$(20.11) \quad \int_{\Xi} \bar{\varphi}_n dx dv = \int_S \nu_n dx dv + \lambda \int_{\partial_h \Xi} \partial_n \bar{\varphi}_n ds + \lambda \int_{\partial_v \Xi \setminus \partial_a \Xi} v \bar{\varphi}_n n_x ds$$

where we denote as $\partial_h \Xi$, $\partial_v \Xi$ the horizontal and vertical parts of the boundary of Ξ respectively.

We remark that deriving this formula we need some careful estimates near the singular points. More precisely, the results of [31] imply that the functions $\bar{\varphi}_n$ are Hölder continuous. A rescaling argument analogous to the one used in [31] implies an estimate with the form:

$$(20.12) \quad |\bar{\varphi}_n| + \left(|x - x_0| + |v|^3 \right)^{\frac{1}{3}} |\partial_v \bar{\varphi}_n| \leq C \left(|x - x_0| + |v|^3 \right)^a$$

for some $a > 0$, where $(x_0, 0)$ is one of the singular points. In order to compute $\int_{\Xi} \bar{\varphi}_n dx dv$ we then approximate the integrals by domains with the form $\int_{\Xi \setminus \bar{\mathcal{R}}_\delta} \bar{\varphi}_n dx dv$, where the domains $\bar{\mathcal{R}}_\delta$ are just translated or reflected versions of the domains \mathcal{R}_δ (cf. (4.27)) and they are chosen in order to exclude the integral near the singular points. Using Gauss's Theorem we obtain boundary terms with the form $\int_{\partial \bar{\mathcal{R}}_\delta} [n_v \partial_v \bar{\varphi}_n + n_x v \bar{\varphi}_n] ds$ which converge to zero as $\delta \rightarrow 0$ due to (20.12). On the other hand $\partial_n \bar{\varphi}_n \leq 0$ on $\partial_h \Xi$, and the definition of $\partial_a \Xi$ implies also that $v \bar{\varphi}_n n_x \leq 0$ on $\partial_v \Xi \setminus \partial_a \Xi$. Therefore, (20.11) yields:

$$\int_{\Xi} \bar{\varphi}_n dx dv \leq \int_S \nu_n dx dv$$

A standard compactness argument then yields that, for a suitable subsequence $\bar{\varphi}_n \rightharpoonup \bar{\varphi}$ where $\bar{\varphi}$ is a weak solution of (20.10). Moreover, $\bar{\varphi}$ is smooth in a neighbourhood of $\partial \Xi$, except near the singular points where we have that $\bar{\varphi}$ is Hölder due to the results of [31]. Multiplying (20.10) by ψ we obtain, using again Gauss's Theorem:

$$\int_{\Xi} (\psi - \lambda \mathcal{L}^* \psi) \bar{\varphi} = \int_S \nu \psi$$

where we use the fact that the contribution near the singular points is zero due to the hölderianity of ψ , $\bar{\varphi}$ and arguing as in the derivation of (20.11). Using (20.8) we obtain

$\int_{\Xi} \zeta \bar{\varphi} = \int_S \nu \psi$, whence $\int_D \bar{\varphi} \geq \int_S \nu \psi$. On the other hand, we can estimate a lower estimate of ψ using (20.9). Indeed, we have:

$$(20.13) \quad \mathbb{P}^{(x,v)} \left(\left\{ \xi_t^{(x,v)} \in \hat{D}, \text{ for } T \subset [0, 1], t \in T, \tau(\Xi) \geq 1 \right\} \right) \geq b_* > 0, \quad (x, v) \in S$$

This inequality can be proved as follows. Suppose that we denote as $G(x, v, t)$ the solution of the problem:

$$(20.14) \quad \partial_t G + \bar{v} \partial_{\bar{x}} G = G_{\bar{v}\bar{v}} \text{ for } t \geq 0, \quad (\bar{x}, \bar{v}) \in \Xi, \quad G = 0 \text{ on } \partial_a^* \Xi$$

$$(20.15) \quad G(\bar{x}, \bar{v}, 0) = \delta_{(x,v)}$$

Then:

$$\mathbb{P}^{(x,v)} \left(\left\{ \xi_{t_*}^{(x,v)} \in A, \xi_t^{(x,v)} \in \Xi \text{ for } t \in [0, t_*] \right\} \right) = \int_A G(\bar{x}, \bar{v}, t_*) d\bar{x} d\bar{v}$$

for any measurable $A \subset \Xi$ and $t_* > 0$. The solvability of (20.14), (20.15) can be proved with some suitable modification of the proof in [31] of the existence of solutions for the Kolmogorov equation in bounded domains with absorbing boundary conditions, or, alternatively, adapting the representation formula (20.9) to this case.

We can obtain a subsolution for (20.14), (20.15) as follows. Suppose that we denote the fundamental solution of the Kolmogorov operator in the plane as $Q(\bar{x}, \bar{v}, t)$. We then obtain a family of subsolutions for (20.14), (20.15) as

$$G_-(\bar{x}, \bar{v}, t; M) = \max \{Q(\bar{x}, \bar{v}, t) - M, 0\}, \text{ for any } M > 0$$

if $0 \leq t \leq \bar{t}_1$ for a suitable $\bar{t}_1 > 0$. More precisely, we will choose M and \bar{t}_1 as follows. We fix $\bar{t}_1 > 0$ arbitrarily small. We then choose $M > 0$ small enough to obtain $Q(\bar{x}, \bar{v}, t) < M$ in $\partial \Xi$ for $0 \leq t \leq \bar{t}_1$. Notice that, since the distance between (x, v) and $\partial \Xi$ is positive and fixed and $Q(\bar{x}, \bar{v}, t) = \frac{C}{t^2} \exp \left(-\frac{3|\bar{x}-x-\frac{v+\bar{v}}{2}t|^2}{t^3} - \frac{(\bar{v}-v)^2}{4t} \right)$, we have that, if we assume that \bar{t}_1

is sufficiently small, there exists a ball $B_\rho(x, v) \subset \Xi$ such that $G_-(\bar{x}, \bar{v}, t_1; M) \geq \delta_M > 0$ if $(\bar{x}, \bar{v}) \in B_\rho(x, v)$. Moreover, the ball can be chosen independent of \bar{t}_1 (even if \bar{t}_1 is arbitrarily small), reducing the value of M , δ_M if needed. We can construct now a new subsolution with the form $\lambda_1 G_-(\bar{x}, \bar{v}, t + \bar{t}_1; M_1)$ with $\lambda_1 > 0$, $M_1 > 0$ and satisfying $\lambda_1 G_-(\bar{x}, \bar{v}, t + \bar{t}_1; M_1) \leq G_-(\bar{x}, \bar{v}, t_1; M)$ in a suitable ball $B_{\rho_1}(x_1, v_1) \subset \Xi$ with (x_1, v_1) closer to the set \hat{D} than (x, v) . Iterating the argument, and using the fact that the size of the balls where the subsolutions are positive can be chosen independent on the times $\bar{t}_1, \bar{t}_2, \dots$ we obtain that there exists $t_* \in [0, \frac{1}{2}]$ and a ball $B_\rho \subset \hat{D}$ such that $G(\bar{x}, \bar{v}, t_*) > \delta > 0$. Therefore $\mathbb{P} \left(\left\{ \xi_{t_*}^{(x,v)} \in B_\rho \right\} \right) > \delta > 0$. We can now estimate $\mathbb{P} \left(\left\{ \xi_t^{(x,v)} \in \hat{D} \text{ for } t \in [t_*, t_* + \varepsilon_0] \right\} \middle| \xi_{t_*}^{(x,v)} \in B_\rho \right)$ for some $\varepsilon_0 > 0$.

To this end we define a function $\tilde{G}(x, v, t)$ solving:

$$(20.16) \quad \partial_t \tilde{G} + \bar{v} \partial_{\bar{x}} \tilde{G} = \tilde{G}_{\bar{v}\bar{v}} \text{ for } t \geq 0, \quad (\bar{x}, \bar{v}) \in \Xi, \quad \tilde{G} = 0 \text{ on } \partial_a^* \hat{D}$$

$$(20.17) \quad \tilde{G}(\bar{x}, \bar{v}, t_*) = \frac{1}{|B_\rho|} \chi_{B_\rho}$$

Therefore $\mathbb{P} \left(\left\{ \xi_t^{(x,v)} \in \hat{D} \text{ for } t \in [t_*, t_* + \varepsilon_0] \right\} \middle| \xi_{t_*}^{(x,v)} \in B_\rho \right) = \tilde{G}(\bar{x}, \bar{v}, t_* + \varepsilon_0)$. Using subsolutions with the form $G_-(\bar{x}, \bar{v}, t; M)$ it then follows that

$$\mathbb{P} \left(\left\{ \xi_t^{(x,v)} \in \hat{D} \text{ for } t \in [t_*, t_* + \varepsilon_0] \right\} \middle| \xi_{t_*}^{(x,v)} \in B_\rho \right) \geq \delta_* > 0$$

Then (20.13) follows.

Then $\psi(x, v) \geq \frac{e^{-\frac{1}{\lambda}}}{\lambda} b_*$ for $(x, v) \in S$. Therefore $\int_D \bar{\varphi} \geq c_* \int_S \nu$. Finally the pointwise estimate $\varphi \geq c_* \int_S \nu$ in the domain $\hat{D} \subset\subset D$ follows from Harnack inequality for Kolmogorov operators (cf. for instance the results in [13], [36]). \square

20.1. Comparison Theorems. We will use the following version of the weak Maximum Principle.

Proposition 20.6. *Suppose that Ξ is one of the admissible domains in Definition 12.1 and that $\varphi \in L_b^\infty(\Xi)$ is a supersolution for the operator $\mathcal{L}(\cdot) + \kappa$ for some $\kappa \geq 0$. Suppose also that $\varphi \geq 0$ in $\partial_a \Xi$, where φ is defined in $\partial_a \Xi$ in the sense of Definition 19.5.*

Then, we have $\varphi(x, v) \geq 0$ for a.e. $(x, v) \in \Xi$. Moreover, there exists $C_ = C_*(\Xi)$, such that*

$$\varphi(x, v) \geq C_* \kappa \quad \text{for any } (x, v) \in D$$

where $d(\Xi)$ is as in (20.7) and $D = \left\{ (x, v) : \text{dist}((x, v), \partial_a \Xi) \geq \frac{d(\Xi)}{4} \right\}$

Moreover, the constant C_* can be chosen uniformly for sets Ξ contained in a compact set of $[0, \infty) \times (-\infty, \infty)$ and have $0 < c_1 \leq d(\Xi) \leq c_2 < \infty$, $0 < c_1 \leq \text{diam}(\Xi) \leq c_2 < \infty$.

Proof. The proof can be made along the same lines as the proof of Lemma 20.4. The problem is linear in κ . Then, we can assume that $\kappa = 1$. We then assume that $\varphi \in L_b^\infty(\Xi)$ is a supersolution for the operator $\mathcal{L}(\cdot) + 1$. We consider any test function $\zeta \geq 0$, $\zeta \in C^\infty$ supported in the set D . We then define ψ as the solution of $-\mathcal{L}^*(\psi) = \zeta$ in Ξ , $\psi = 0$ on $\partial_a^* \Xi$. We have the representation formula:

$$\psi(x, v) = \mathbb{E}^{(x, v)} \left(\psi \left(\xi_{\tau(\Xi)}^{(x, v)} \right) \right) + \mathbb{E}^{(x, v)} \left(\int_0^{\tau(\Xi)} \zeta \left(\xi_t^{(x, v)} \right) dt \right), \quad (x, v) \in \Xi$$

with $\xi_t^{(x, v)}$ as in the Proof of Lemma 20.4. Since the trajectories of the process $\xi_t^{(x, v)}$ leave the domain across $\partial_a^* \Xi$ we obtain the first term vanishes. Therefore:

$$\psi(x, v) = \mathbb{E}^{(x, v)} \left(\int_0^{\tau(\Xi)} \zeta \left(\xi_t^{(x, v)} \right) dt \right) \geq 0$$

Using ψ as a test function in the definition of supersolution in Definition 19.2 we obtain $-\int \mathcal{L}^*(\psi) \varphi \geq \int \psi$, whence $\int \zeta \varphi \geq \int \psi$. We now claim that $\int \psi \geq c_* \int \zeta$. To this end we first notice that

$$\psi(x, v) \geq \mathbb{E}^{(x, v)} \left(1_{A(x, v)} \int_0^1 \zeta \left(\xi_t^{(x, v)} \right) dt \right)$$

where $A(x, v)$ is the set of realizations of the Markov process $\xi_t^{(x, v)}$ satisfying $\tau(\Xi) > 1$ and $1_{A(x, v)}$ is the characteristic function of such a set. Then:

$$\psi(x, v) \geq \int_0^1 \mathbb{E}^{(x, v)} \left(1_{A(x, v)} \zeta \left(\xi_t^{(x, v)} \right) \right) dt$$

Notice that we can then obtain the lower estimate

$$\mathbb{E}^{(x, v)} \left(1_{A(x, v)} \zeta \left(\xi_t^{(x, v)} \right) \right) \geq G(x, v, t), \quad 0 \leq t \leq 1$$

and where G is the solution of the problem:

$$\begin{aligned} \partial_t G + \bar{v} \partial_{\bar{x}} G &= G_{\bar{v} \bar{v}} \quad \text{for } t \geq 0, \quad (\bar{x}, \bar{v}) \in \Xi, \quad G = 0 \text{ on } \partial_a^* \Xi \\ G(\bar{x}, \bar{v}, 0) &= \zeta(\bar{x}, \bar{v}) \end{aligned}$$

Using then that ζ is supported in D we can argue as in the Proof of Lemma 20.4, using suitable subsolutions for the fundamental solution of the problem above to obtain $\psi(x, v) \geq$

$G(x, v, 0) \geq c_0 \int \zeta$. Then $\int \zeta \varphi \geq c_* \int \zeta$ and since ζ is arbitrary it follows that $\varphi \geq c_*$. Using the linearity of the problem the result would follow. \square

We will need also the following version of the strong maximum principle.

Proposition 20.7. (*Strong maximum principle*) Suppose that Ξ is one admissible domain in the sense of Definition 12.1. Let $\Phi \in C(\Xi)$, $\Phi \geq 0$ and let $\mathcal{L}\Phi = 0$ on $\Phi \in C(\Xi)$ with Φ satisfying (11.1) if $\bar{\Xi} \cap \{(x, v) = (0, v), v \in \mathbb{R}\} \neq \emptyset$. Suppose that there exists a point $(\hat{x}, \hat{v}) \in \bar{\Xi} \setminus \partial_a \Xi$ such that $\Phi(\hat{x}, \hat{v}) = 0$. Then $\Phi \equiv 0$.

In order to prove Proposition 20.7 we will use the following Lemma.

Lemma 20.8. Let the function $G(v, x) \in C([1, 2] \times [0, \infty))$ satisfy the equation

$$\begin{aligned} vG_x &= G_{vv}, \quad 1 \leq v \leq 2, \quad x > 0, \\ G(v, 0) &\geq 1 \text{ for } 7/4 \leq v \leq 2. \\ G(7/4, x) &\geq 0 \text{ for } x \geq 0 \end{aligned}$$

Then there exists $\beta > 0$ not depending on G , such that $G(v, x) \geq \beta > 0$ for $5/4 \leq v \leq 7/4$, $1 \leq x \leq 2$.

Proof. We prove this lemma by constructing a sub-solution \bar{G} . Let $\bar{G}(v, x) = e^{-\theta x} \sin(4\pi(v - 7/4))$. Then for $7/4 \leq v \leq 2$, \bar{G} is a sub-solution of G if we choose $\theta > 0$ in such a way that $\theta \geq 64\pi^2/7$. Since $G(v, 0) \geq 1 \geq \bar{G}(v, 0)$, for $7/4 \leq v \leq 2$ and $G(v, x) \geq 0 = \bar{G}(v, x)$ at $v = 7/4, 2$ for all $x > 0$. Thus we have $G(v, x) \geq \bar{G}(v, x)$ for $7/4 \leq v \leq 2$. Let $Q = [1, 2] \times [0, 1]$ and $\tilde{Q} = [5/4, 7/4] \times [1, 2]$, then we get

$$\sup_Q G(v, x) \geq e^{-\theta} > 0.$$

We then apply the classical Harnack inequality for parabolic equations (cf. for instance [18]) to obtain

$$\inf_{\tilde{Q}} G(v, x) \geq C \sup_Q G(v, x) \geq Ce^{-\theta} \equiv \beta > 0.$$

This completes the proof. \square

Proof of Proposition 20.7. We prove by contradiction that if there exists a point at which Φ is positive, then Φ is positive everywhere. This implies the Proposition. Suppose that there exists $(x^*, v^*) \in [0, \infty) \times (-\infty, \infty) \setminus (0, 0)$ with $\Phi(x^*, v^*) > 0$. $\mathcal{L}\Phi = 0$ reads

$$(20.18) \quad -v\Phi_x = \Phi_{vv}$$

We may assume that $v^* > 0$ since the cases $v^* < 0$ can be treated similarly and if $v^* = 0$, then we can find $v^{**} > 0$ such that $\Phi(x^*, v^{**}) > 0$ due to the continuity of Φ . Our goal is to show that $\Phi(x, v) > 0$ for all $(x, v) \in [0, \infty) \times (-\infty, \infty) \setminus (0, 0)$.

First we prove the positivity of Φ for $0 \leq x < x^*, v > 0$. The equation (20.18) is a parabolic equation for $v \geq \rho > 0$, ρ arbitrary and thus $\Phi(x, v) > 0$ for $0 \leq x < x^*, v > 0$ due to the strong maximum principle for non-degenerate parabolic equations (cf. [18]).

Next we prove the positivity for $0 \leq x < x^*, v = 0$. We rescale as follows. For $x \in [x_0, x_0 + 2^{-n}]$, $v \in [2^{-(n+1)/3}, 2^{(2-n)/3}]$, define for some $x_0 \in (0, x^*)$,

$$\bar{x} = -2^{(n+1)} [x - (x_0 + 2^{-n})], \quad \bar{v} = 2^{(n+1)/3} v,$$

$$G_n(\bar{v}, \bar{x}) = \frac{\Phi(x, v)}{M_n}, \quad M_n = \inf_{v \in I_n} \Phi(x_0 + 2^{-n}, v), \quad I_n = \left[\frac{7}{4} 2^{-(n+1)/3}, 2 \cdot 2^{-(n+1)/3} \right]$$

Then the following holds

$$\bar{v} \partial_{\bar{x}} G_n = \partial_{\bar{v}}^2 G_n, \quad \bar{x} \in [0, 2], \quad \bar{v} \in [1, 2],$$

$$G_n(\bar{v}, 0) \geq 1 \text{ for } 7/4 \leq \bar{v} \leq 2.$$

If we let $J_n = [\frac{5}{4}2^{-(n+1)/3}, \frac{7}{4}2^{-(n+1)/3}]$, then $I_{n+1} \subset J_n$ for each n . We now apply Lemma 20.8 to get

$$G_n(\bar{v}, \bar{x}) \geq \beta > 0, \text{ for } \bar{v} \in [5/4, 7/5], \quad \bar{x} \in [1, 2],$$

or equivalently,

$$(20.19) \quad \Phi(x, v) \geq \beta M_n, \text{ for } (x, v) \in [x_0, x_0 + 2^{-(n+1)}] \times J_n.$$

In particular, at $\bar{x} = 1$,

$$\frac{\Phi(x_0 + 2^{-(n+1)}, v)}{M_n} \geq \beta, \text{ for } v \in J_n.$$

Thus we get, since $I_{n+1} \subset J_n$,

$$M_{n+1} = \inf_{v \in I_{n+1}} \Phi(x_0 + 2^{-(n+1)}, v) \geq \beta M_n,$$

which yields

$$M_n \geq C\beta^n, \quad 0 < \beta < 1$$

Then from (20.19), we have for each n ,

$$(20.20) \quad \Phi(x_0, v) \geq C\beta^{n+1}, \text{ for } v \in J_{n+1}.$$

Now we can rewrite (20.20) as

$$\begin{aligned} \Phi(x_0, v) &\geq C\beta^{n+1} = C \exp\left(-\frac{\log(2)}{3}(n+1)\right)^{-\frac{3\log(\beta)}{\log(2)}} \\ &\geq Cv^\gamma, \quad \gamma = -\frac{2\log(\beta)}{\log(2)} > 0, \text{ for } 0 < v < 2. \end{aligned}$$

Now let $0 < \tilde{x} < x^*$, then the argument above yields, for $x_0 \in (\tilde{x}, \tilde{x} + \delta)$ with $\delta > 0$ small,

$$\Phi(x_0, v) \geq Cv^\gamma, \text{ for } \gamma > 0.$$

We first construct a sub-solution $G(\bar{v}, \bar{x}) = (\bar{x})^{\gamma/3} H\left(\frac{\bar{v}}{(\bar{x})^{1/3}}\right)$ satisfying

$$(20.21) \quad \bar{v}G_x = G_{\bar{v}\bar{v}}, \quad G(\bar{v}, 0) = (\bar{v})^\gamma$$

where $H(\xi)$, $\xi = \frac{\bar{v}}{(\bar{x})^{1/3}}$ satisfy:

$$(20.22) \quad \frac{\gamma}{3}\xi H - \frac{\xi^2}{3}H_\xi = H_{\xi\xi}, \quad H(0) = 0, H_\xi(0) > 0, \quad H(\xi) \sim \xi^\gamma, \text{ as } \xi \rightarrow \infty.$$

The existence of a solution \bar{H} of the differential equation in (20.22) with $\bar{H}(0) = 0$, $\bar{H}_\xi(0) = 1$ follows from standard ODE theory. Then we can show that $\bar{H}_\xi(\xi) > 0$, for all $\xi \geq 0$, because otherwise, let ξ^* be the first value where $\bar{H}_\xi(\xi^*) = 0$, then $\bar{H}(\xi^*) > 0$ and $\bar{H}_{\xi\xi}(\xi^*) = \frac{\gamma}{3}\xi^*\bar{H}(\xi^*) > 0$, which leads to a contradiction. The asymptotic behaviors of the solutions of the differential equation in (20.22) can be computed using the results in Chapter 13 of [1], whence:

$$\bar{H}(\xi) \sim C\xi^\gamma, \quad \xi \rightarrow \infty \text{ with } C > 0$$

Then $G(\bar{v}, \bar{x}) = (\bar{x})^{\gamma/3} H\left(\frac{\bar{v}}{(\bar{x})^{1/3}}\right)$ satisfies (20.21) and for all $\bar{x} > 0$,

$$G_{\bar{v}}(0^+, \bar{x}) = (\bar{x})^{\gamma/3-1/3} H_\xi(0) > 0.$$

By a comparison principle and Taylor expansion, there exist constant C_0 and C_1 satisfying, for $x \in [\tilde{x}, \tilde{x} + \delta)$,

$$\Phi(x, v) \geq C_0 G(v, x - (\tilde{x} + \delta)) \geq C_1 (\tilde{x} + \delta - x)^{\frac{\gamma-1}{3}} v, \text{ for } v \text{ small.}$$

This implies $\Phi(x, 0) > 0$ for $x \in [\tilde{x}, \tilde{x} + \delta)$, since otherwise, $\Phi(x, v) < 0$ for some $v < 0$, a contradiction. In particular, we obtain $\Phi(\tilde{x}, 0) > 0$ for any $0 < \tilde{x} < x^*$. Finally, the boundary condition (11.1) for Φ and similar arguments applying the forward parabolic equation $-v\Phi_x = \Phi_{vv}$, $v < 0$ yield that $\Phi(x, v) > 0$ for all $(x, v) \neq (0, 0)$. This contradicts the assumption $\Phi(\hat{x}, \hat{v}) = 0$ and the result follows. \square

We will use also comparison arguments in a class of domains whose boundary contains the singular point. We define the following class of domains:

$$(20.23) \quad \Lambda_R = \left\{ (x, v) : 0 < x < R, -R^{\frac{1}{3}} < v < rR^{\frac{1}{3}} \right\}, \quad R > 0$$

as well as the admissible boundary:

$$\partial_a \Lambda_R = \left((0, R) \times \left\{ -R^{\frac{1}{3}} \right\} \right) \cup \left((0, R) \times \left\{ rR^{\frac{1}{3}} \right\} \right) \cup \left(\{R\} \times (0, rR^{\frac{1}{3}}) \right)$$

We have the following result.

Proposition 20.9. *Suppose that $\varphi \in C(\overline{\Lambda_R})$ satisfies $\mathcal{L}\varphi = 0$ in Y in the sense of Definition 11.7 as well as the compatibility condition (11.1), $\varphi \geq 0$ on $\partial_a \Lambda_R$. If $r < r_c$ we will assume in addition that $\varphi(0, 0) = 0$. Then $\varphi \geq 0$ in Λ_R .*

Proof. We can apply the comparison result Proposition 20.6 in the domain $\Lambda_R \setminus \mathcal{R}_\delta$. Applying then Proposition 20.6 to the function $\tilde{\varphi} + \varepsilon$ we obtain $\varphi \geq -\varepsilon$ in $\Lambda_R \setminus \mathcal{R}_\delta$. Taking the limit $\varepsilon \rightarrow 0$ and $\delta \rightarrow 0$ we obtain the result. \square

21. ASYMPTOTIC BEHAVIOUR OF THE SOLUTIONS OF SOME PDES NEAR THE SINGULAR POINT.

In several of the arguments we will need detailed information about the asymptotics of the solutions of the problem $\mathcal{L}\varphi = h$, where $\varphi, h \in C(X)$ and $\mathcal{L}\varphi$ is as in Definition 11.7. We now prove several results which will be used later in order to derive such information.

Lemma 21.1. *Suppose that $0 < r \leq 1$. There exists a function $S \in C^2(\mathcal{U}) \cap C(\overline{\mathcal{U}})$ such that*

$$(21.1) \quad \mathcal{L}S = 1 \text{ in } \mathcal{U}$$

$$(21.2) \quad S(0, -v) = S(0, rv) \text{ for all } v > 0$$

$$(21.3) \quad S = O\left(x^{\frac{2}{3}} + |v|^2\right) \text{ as } (x, v) \rightarrow 0$$

Moreover, if $r = 1$ we have $S(x, v) = \frac{v^2}{2}$. If $r < 1$, the function $S(x, v)$ takes positive and negative values in $\mathbb{R}^+ \times \mathbb{R}$.

Proof. We look for S of the following form

$$(21.4) \quad S = x^{2/3} Q\left(\frac{v}{x^{1/3}}\right)$$

Since $\partial_x S = \frac{2}{3}x^{-1/3}Q - \frac{1}{3}vx^{-2/3}Q'$ and $\partial_{vv}S = Q''$, $\partial_v^2 S + v\partial_x S = 1$, the equation (21.1) yields:

$$(21.5) \quad Q'' - \frac{1}{3}z^2 Q' + \frac{2}{3}zQ = 1.$$

This equation has an explicit solution:

$$Q_p(z) = \frac{z^2}{2}$$

On the other hand we recall that the homogeneous solution of (21.5) can be obtained by means of Kummer functions. We write $\Psi(z) = Q(z) - Q_p(z)$. Then:

$$(21.6) \quad \Psi'' - \frac{1}{3}z^2\Psi' + \frac{2}{3}z\Psi = 0$$

It is readily seen that the solution of (21.6) is given by:

$$\Psi(z) = c_1 M\left(-\frac{2}{3}, \frac{2}{3}; \frac{z^3}{9}\right) + c_2 U\left(-\frac{2}{3}, \frac{2}{3}; \frac{z^3}{9}\right)$$

with $c_1, c_2 \in \mathbb{R}$. In order to obtain (21.3) we must choose $c_1 = 0$, in order to avoid the exponential growth of $\Psi(z)$. We notice that, since $U\left(-\frac{2}{3}, \frac{2}{3}; \frac{z^3}{9}\right)$ solves (21.6), it is analytic in the complex plane $z \in \mathbb{C}$. Using Proposition 4.2, and more precisely Remark 4.6, we obtain:

$$\begin{aligned} S(0, v) &= \left(\frac{1}{2} + c_2\right)v^2, \quad v > 0 \\ S(0, v) &= \left(\frac{1}{2} - 2c_2\right)v^2, \quad v < 0 \end{aligned}$$

where we use that $K_{\frac{2}{3}} = 2 \cos(\pi) = -2$. We then impose the boundary condition $S(0, -v) = S(0, rv)$ for $v > 0$ (cf. (21.2)), whence:

$$\left(\frac{1}{2} - 2c_2\right) = \left(\frac{1}{2} + c_2\right)r^2$$

Therefore $c_2 = \frac{(1-r^2)}{2(2+r^2)}$. It then follows that

$$Q(z) = \frac{z^2}{2} + \frac{(1-r^2)}{2(2+r^2)} U\left(-\frac{2}{3}, \frac{2}{3}; \frac{z^3}{9}\right)$$

Notice that, if $r = 1$ we obtain $Q(z) = \frac{z^2}{2}$. If $r \neq 1$ we obtain, using 13.5.10 in [1] that:

$$(21.7) \quad Q(0) = \frac{(1-r^2)\Gamma\left(\frac{1}{3}\right)}{2(2+r^2)\Gamma\left(-\frac{1}{3}\right)}$$

Then $Q(0) < 0$ if $r < 1$. On the other hand, using that $U\left(-\frac{2}{3}, \frac{2}{3}; \xi\right) \sim \xi^{\frac{2}{3}}$ as $\xi \rightarrow \infty$ it follows that $Q(z) \sim \left(\frac{1}{2} + \frac{(1-r^2)}{2(2+r^2)9^{\frac{2}{3}}}\right)z^2$ as $z \rightarrow \infty$. Therefore $Q(z) > 0$ for large values of $|z|$ and there exists a range of values for which $Q(z) < 0$. \square

We now prove some Liouville's type Theorems which will be useful deriving asymptotic formulas near the singular point of the solutions of equations like $\lambda\psi - \mathcal{L}\psi = g$ with boundary conditions $\psi(0, rv) = \psi(0, -v)$, $v > 0$.

Theorem 21.2. (i) Let $0 < r < r_c$. Suppose that ψ is a classical solution of $\mathcal{L}\psi = 0$ in $\{x > 0, v \in \mathbb{R}\}$ satisfying $\psi(0, rv) = \psi(0, -v)$, $v > 0$ as well as the estimate $|\psi(x, v)| \leq C(|x| + |v|^3)^b$ for some $b \in (0, \beta) \cup (\beta, \frac{2}{3}]$ and $(x, v) \in \{x > 0, v \in \mathbb{R}\}$. Then $\psi = 0$.

(ii) There is not any bounded function ψ defined in \mathbb{R}^2 which satisfies also one equation with the form $\tilde{\mathcal{L}}\psi = 1$, where $\tilde{\mathcal{L}}$ is one of the operators in the set

$$\{D_v^2 + vD_x, D_v^2 + D_x, D_v^2 - D_x\}.$$

(iii) Let $r > r_c$. Suppose that we have a solution of $\mathcal{L}\psi = 0$ in $\{x > 0, v \in \mathbb{R}\}$ satisfying $\psi(0, rv) = \psi(0, -v)$, $v > 0$ as well as the estimate $|\psi(x, v)| \leq C \left(|x| + |v|^3\right)^b$ with $0 < b \leq \frac{2}{3}$. Then $\psi = 0$.

(iv) Let $0 < r < r_c$. Suppose that ψ is a classical solution of $\mathcal{L}\psi = 0$ in $\{x > 0, v \in \mathbb{R}\}$ satisfying $\psi(0, rv) = \psi(0, -v)$, $v > 0$ as well as the estimate $|\psi(x, v)| \leq C$ for $(x, v) \in \{x > 0, v \in \mathbb{R}\}$. Then $\psi = K_0$ for some $K_0 \in \mathbb{R}$.

The proof of Theorem 21.2 will require to prove some previous auxiliary results, which will be required in order to deal with the cases (i), (iii).

We first prove the following result:

Proposition 21.3. Suppose that φ solves $\mathcal{L}(\varphi) = 0$ in $x \geq 0, v \in \mathbb{R}$ and satisfies the boundary conditions $\varphi(0, rv) = \varphi(0, -v)$, $v > 0$. Suppose also that we have the estimates:

$$(21.8) \quad |\varphi(x, v)| \leq K \left(|x| + |v|^3\right)^b \quad \text{with } 0 \leq b \leq \frac{2}{3} \quad \text{in } x \geq 0, v \in \mathbb{R}$$

$$(21.9) \quad \sup_{(|x|+|v|^3)=1} |\varphi(x, v)| = 1$$

for some $K > 1$. Then $b = \beta$ or $b = 0$.

This proposition will be proved with the help of the following result.

Proposition 21.4. Suppose that φ solves $\mathcal{L}(\varphi) = 0$ in $x \geq 0, v \in \mathbb{R}$ and satisfies the boundary conditions $\varphi(0, rv) = \varphi(0, -v)$, $v > 0$. Suppose also that φ satisfies

$$(21.10) \quad \varphi(\mu^3 x, \mu v) = \mu^{3b} \varphi(x, v)$$

with $0 \leq b \leq \frac{2}{3}$. Then, if $0 < r < r_c$ we have that $\varphi = 0$ unless $b = \beta$ or $b = 0$. If $r > r_c$ we have $\varphi = 0$ unless $b = 0$.

It is convenient to introduce a different group of variables in order to simplify the homogeneity of the functions. We define $x = \zeta^3$. Then $\mathcal{L}(\varphi) = 0$ if and only if:

$$(21.11) \quad 3\zeta^2 \varphi_{vv} + v \varphi_\zeta = 0, \quad \varphi(0, rv) = \varphi(0, -v), \quad v > 0$$

In order to prove Proposition 21.3 we will use the following auxiliary result.

Lemma 21.5. Let us denote as \mathcal{Y} the set of functions with the form $G : \mathcal{U} \times \mathbb{R} \rightarrow \mathbb{C}$ satisfying the following properties:

- (i) $G(X, \omega)$ satisfies the homogeneity condition $G(\mu X, \omega) = G(X, \omega)$ for any $\mu > 0$.
- (ii) The support of $G(X, \omega)$ is contained in the set $K \times [-M, M]$ where $K \subset \mathcal{U}$ is a compact set and $0 < M < \infty$.
- (iii) $G \in C^\infty(\mathcal{U} \times \mathbb{R})$.

Then, the set of functions with the form:

$$(21.12) \quad F(\zeta, v) = F(X) = \int_{\mathbb{R}} |X|^{i\omega} G(X, \omega) d\omega, \quad X = (\zeta, v)$$

is dense in $L^1\left(\mathcal{U}; \frac{dX}{|X|^2}\right)$ where $dX = d\zeta dv$.

Proof. Since G is homogeneous, the representation formula (21.12) is equivalent to the representation of the function $f(r, \theta) = F(X)$ with $r = |X|$ and $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ as angular coordinate, in terms of the Mellin transform. Moreover, using the change of variables $\tilde{f}(t, \theta) = f(r, \theta)$, $t = \log(r)$ and writing $G(X, \omega) = g(\omega, \theta)$ the problem becomes equivalent to proving that the class of functions:

$$(21.13) \quad \tilde{f}(t, \theta) = \int_{\mathbb{R}} e^{i\omega t} g(\omega, \theta) d\omega$$

with $g \in C_c^\infty(\mathbb{R} \times [-\frac{\pi}{2}, \frac{\pi}{2}])$ is dense in $L^1(\mathbb{R} \times [-\frac{\pi}{2}, \frac{\pi}{2}])$. This follows using first that the compactly supported functions are dense in $L^1(\mathbb{R} \times [-\frac{\pi}{2}, \frac{\pi}{2}])$ and using then that the functions with the form (21.13) are dense in $L^1(J)$ for any compact set $J \subset \mathbb{R} \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. \square

The next Lemma shows, by means of an explicit computation, that there are not homogeneous solutions of (21.11) for a set of homogeneities and values of r .

Lemma 21.6. *Suppose that φ satisfies (21.11) and that φ satisfies the homogeneity condition $\varphi(\mu X) = \mu^{3b} \varphi(X)$, $X = (\zeta, v) \in \mathcal{U}$, for any $\mu > 0$ where $\operatorname{Re}(b) \in [0, \beta) \cup (\beta, \frac{2}{3}]$ if $0 < r < r_c$ and $0 \leq \operatorname{Re}(b) \leq \frac{2}{3}$ if $r_c < r \leq 1$. Then $\varphi(\zeta, v) \equiv 0$ unless $b = 0$.*

Remark 21.7. *In the case $r_c < r$ it is possible to prove the Lemma for $r_c < r \leq \sqrt{2}$. Notice that, if $\operatorname{Im}(b) \neq 0$ we must necessarily have φ complex.*

Proof. Using the change of variables $x = \zeta^3$, it follows that the solutions with the homogeneity indicated in Lemma 21.6 have the form (10.4). Replacing β by z to avoid ambiguities, we would obtain that $\varphi = \varphi(x, v)$ has the form $\varphi(x, v) = x^z F\left(\frac{v^3}{9x}\right)$ for some $z \in \mathbb{C}$ where $\operatorname{Re}(z) \in [0, \frac{2}{3}]$. Our goal is to show that the only possible values of z are $z = \beta(r)$ and $z = 0$ if $0 < r < r_c$ and $z = 0$ if $r_c < r \leq 1$. To this end we argue as in the Proof of Proposition 10.2 to prove that

$$(21.14) \quad r^{3z} = 2 \sin \left(\pi \left(\frac{1}{6} - z \right) \right)$$

We then need to show that (21.14) has only the roots indicated above. To this end we argue as follows. If $r > 0$ is sufficiently small the only roots of (21.14) in the strip $0 \leq \operatorname{Re}(z) \leq \frac{2}{3}$ are at $z = 0$ and $z_*(r)$ in a neighbourhood of $z = \frac{1}{6}$. Indeed, it follows from the Implicit Functions Theorem that for any $\delta > 0$ there is only one solution of (21.14) in the region $\operatorname{Re}(z) \geq \delta > 0$ if r is small, and such root converges to $z = \frac{1}{6}$ as $r \rightarrow 0$. In order to study the possible roots with $\operatorname{Re}(z) < \delta$ we write $z = A + iB$, $A, B \in \mathbb{R}$ with $0 \leq A < \delta$. Then (21.14) becomes:

$$(21.15) \quad \begin{aligned} & e^{3A \log(r)} (\cos(3B \log(r)) + i \sin(3B \log(r))) \\ &= 2 \sin \left(\pi \left(\frac{1}{6} - A \right) \right) \cosh(\pi B) - 2i \cos \left(\pi \left(\frac{1}{6} - A \right) \right) \sinh(\pi B) \end{aligned}$$

Estimating $\cos(3B \log(r))$ and $\cosh(\pi B)$ it follows that the real part of the left-hand side of (21.15) is smaller than $e^{3A \log(r)}$ and the real part of the left hand side of (21.15) is larger than $2 \sin(\pi(\frac{1}{6} - A))$. It follows, using Taylor's Theorem to approximate this functions for $A > 0$ small that $2 \sin(\pi(\frac{1}{6} - A)) \geq 1 - C_0 A$ for some constant $C_0 > 0$ independent of r . Since $e^{3A \log(r)} < 1 - C_0 A$ for r small enough it follows that the only solution of (21.15) with $A < \delta$ has $A = 0$ if r is small. Then (21.15) becomes:

$$\cos(3B \log(r)) + i \sin(3B \log(r)) = \cosh(\pi B) - i\sqrt{3} \sinh(\pi B)$$

and using the fact that $\cos(3B \log(r)) \leq 1 < \cosh(\pi B)$ if $B > 0$ we obtain that $B = 0$.

Notice that for any fixed $r < 1$ we have $|2 \sin(\pi(\frac{1}{6} - z))| > |r^{3z}|$ if z is large enough. Standard properties of the zeros of analytic functions then imply that the number of solutions of (21.14) for each $r > 0$ in $0 \leq \operatorname{Re}(z) \leq \frac{2}{3}$ can change for the values of r for which there are solutions of (21.14) in $\operatorname{Re}(z) = 0$ or $\operatorname{Re}(z) = \frac{2}{3}$. Writing $z = iB$, $B \in \mathbb{R}$ we obtain that (21.14) becomes (cf. (21.15)):

$$\cos(3B \log(r)) + i \sin(3B \log(r)) = \cosh(\pi B) - i\sqrt{3} \sinh(\pi B)$$

and using again that $\cos(3B \log(r)) \leq 1 < \cosh(\pi B)$ it follows that there are not solutions of (21.14) with $\operatorname{Re}(z) = 0$ except $z = 0$. We now prove that there are not solutions of (21.14) neither with $\operatorname{Re}(z) = \frac{2}{3}$. To this end we write $z = \frac{2}{3} + iB$, $B \in \mathbb{R}$ and (21.14) becomes:

$$r^2 (\cos(3B \log(r)) + i \sin(3B \log(r))) = -2 \cosh(\pi B)$$

The solutions of this equation satisfy $\sin(3B \log(r)) = 0$, whence $3B \log(r) = n\pi$, $n \in \mathbb{Z}$. Then $r^2 \cos(n\pi) = -2 \cosh\left(\frac{n\pi^2}{3 \log(r)}\right)$, whence $n = 2\ell + 1$, $\ell \in \mathbb{Z}$ and $r^2 = 2 \cosh\left(\frac{(2\ell+1)\pi^2}{3 \log(r)}\right)$. This equation does not have any solution with $r \leq \sqrt{2}$. It then follows that the number of roots of (21.14) in the strip $0 \leq \operatorname{Re}(z) \leq \frac{2}{3}$ can change only if a multiple root is formed at $z = 0$ for some value of r . A simple computation shows that this happens only for $r = r_c$, in which case there is a double root (21.14) at $z = 0$. Therefore there is one root of (21.14) in $0 < \operatorname{Re}(z) \leq \frac{2}{3}$ if $r < r_c$ and no roots if $r_c < r \leq 1$, whence the result follows. \square

Proposition 21.4 then follows from Lemma 21.6.

We will use the following standard abstract result of Functional Analysis, which is just an adaptation of Fredholm alternative adapted to our specific situation.

Proposition 21.8. *Suppose that X is a reflexive Banach space. Let $A : D(A) \subset X \rightarrow X$ be a linear closed operator with domain $D(A)$ dense in X . Let A^* be the adjoint of A , with domain $D(A^*) \subset X^*$, with $D(A^*)$ dense in X^* . If $\operatorname{Ker}(A) = 0$, then $R(D(A^*))$ is dense in X^* .*

Proof. Suppose that $R(D(A^*))$ is not dense in X^* . Then $\overline{R(D(A^*))} \neq X^*$. Therefore, Hahn-Banach implies the existence of an element $\ell \in X^{**}$ such that $\langle \ell, \eta \rangle = 0$ for any $\eta \in \overline{R(D(A^*))}$ and with $\ell \neq 0$. Since X is reflexive we have $X^{**} = X$. Then, there exist $x_0 \in X$, $x_0 \neq 0$ such that $\langle \eta, x_0 \rangle = 0$ for any $\eta \in \overline{R(D(A^*))}$. In particular, given any $z \in D(A^*)$ we have $0 = \langle A^* z, x_0 \rangle$. Therefore, using the definition of adjoint operator, it follows that $x_0 \in D(A^{**})$ and $A^{**} x_0 = 0$. Theorem 3.24 in [10] implies $A^{**} = A$. Then $A x_0 = 0$. Since $\operatorname{Ker}(A) = 0$ we have $x_0 = 0$ and this gives a contradiction, whence $R(D(A^*))$ is dense in X^* . \square

We can use now Lemma 21.5 to show that the solutions of (21.11) satisfying a suitable boundedness condition are homogeneous. It then follows, using Lemma 21.6 that they vanish for suitable ranges of parameters. More precisely, we have the following result.

Lemma 21.9. *Suppose that $\varphi = \varphi(\zeta, v)$ satisfies (21.11) as well as the inequality:*

$$|\varphi(\zeta, v)| \leq K(|\zeta| + |v|)^{3b} \text{ in } x \geq 0, \quad v \in \mathbb{R}, \quad K < \infty$$

with $\operatorname{Re}(b) \in [0, \beta) \cup (\beta, \frac{2}{3}]$ if $0 < r < r_c$ and $0 \leq b \leq \frac{2}{3}$ if $r_c < r \leq 1$. Then $\varphi(\zeta, v) \equiv 0$ unless $b = 0$.

Proof. We multiply (21.11) by a C^∞ test function $F(\zeta, v)$ with compact support in \mathcal{U} and satisfying

$$(21.16) \quad F(0, v) = r^{-2} F\left(0, -\frac{v}{r}\right), \quad v > 0$$

Integrating by parts we obtain, with the help of (21.16) and using also that $\varphi(0, rv) = \varphi(0, -v)$ for $v > 0$ we obtain:

$$(21.17) \quad \int_{\mathcal{U}} \mathcal{B}^* F d\zeta dv = 0$$

where from now on, we write, by shortness:

$$\begin{aligned} \mathcal{B} &= 3\zeta^2 \partial_{vv} + v \partial_\zeta \\ \mathcal{B}^* &= 3\zeta^2 \partial_{vv} - v \partial_\zeta \end{aligned}$$

We take $F(\zeta, v)$ with the following form

$$F(\zeta, v) = F(X) = \int_{\mathbb{R}} |X|^{\alpha+i\omega} G(X, \omega) d\omega, \quad X = (\zeta, v)$$

with $G(X, \omega)$ to be specified. Then (21.17) becomes

$$(21.18) \quad \int_{\mathbb{R}} \int_{\mathcal{U}} \mathcal{B}^* \left(|X|^{\alpha+i\omega} G(X, \omega) \right) \varphi d\zeta dv d\omega = 0$$

Notice that if G satisfies:

$$(21.19) \quad G(0, v, \omega) = \frac{1}{|r|^{i\omega+2+\alpha}} G\left(0, -\frac{v}{r}, \omega\right), \quad v > 0$$

we obtain that F satisfies (21.16).

We choose $\alpha = -3b - 2$. Then $g(X) = |X|^{\alpha+2} \varphi(X)$ is globally bounded by the assumption on φ . We write:

$$(21.20) \quad \begin{aligned} \eta(X, \omega) &= \frac{1}{|X|^{\alpha+i\omega}} \mathcal{B}^* \left(|X|^{\alpha+i\omega} G(X, \omega) \right) \\ \xi(X) &= \int_{\mathbb{R}} |X|^{i\omega} \eta(X, \omega) d\omega \end{aligned}$$

Then, (21.18) implies:

$$(21.21) \quad \int_{\mathcal{U}} \xi(X) g(X) \frac{dX}{|X|^2} = 0$$

for any function $\xi(\cdot)$ with the form (21.20). Our goal is to show that if $b \leq 0$, the functions ξ in (21.20) can be chosen in a dense set in $L^1\left(\mathcal{U}; \frac{dX}{|X|^2}\right)$, which will imply $g(X) = 0$ and hence $\varphi = 0$. To this end, we define the following operators for each $\omega \in \mathbb{R}$:

$$(21.22) \quad \mathcal{A}^*(\omega) = \frac{1}{|X|^{\alpha+2}} \mathcal{B}^* \left(|X|^{\alpha+i\omega} G(X, \omega) \right)$$

To make precise the definition of the operators \mathcal{A}^* we need to specify their domain. To this end we define a curve in parametric coordinates by means of $\{r = \lambda(\theta) : \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]\}$, with $\lambda(-\frac{\pi}{2}) = \frac{1}{r}$, $\lambda(\frac{\pi}{2}) = 1$, $\lambda'(\theta) \leq 0$. We then define domains

$$\mathcal{D} = \left\{ \lambda(\theta) < |X| < 2\lambda(\theta) : \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \right\}.$$

The operator $\mathcal{A}^*(\omega)$ in (21.22) will be defined in the space of functions:

$$Y = \{G \in L^2(\mathcal{D}) : G(\mu X) = G(X) \text{ for } \mu > 0, \quad X, \mu X \in \mathcal{D}\}$$

with domain:

$$D(\mathcal{A}^*(\omega)) = \{G \in Y, \partial_v G, \partial_v^2 G, \partial_\zeta G \in L^2(\mathcal{D}), \text{ (21.19) holds}\}$$

Note that the boundary condition (21.19) in the sense of traces is meaningful due to the regularity of the functions in $D(\mathcal{A}^*(\omega))$. The adjoint of $\mathcal{A}^*(\omega)$ is given by the operator:

$$(21.23) \quad \mathcal{A}(\omega) = |X|^{\alpha+i\omega} \mathcal{B} \left(\frac{\tilde{G}(X, \omega)}{|X|^{\alpha+2}} \right)$$

with the boundary condition

$$(21.24) \quad \tilde{G}(0, v, \omega) = r^{\alpha+2} \tilde{G}\left(0, -\frac{v}{r}, \omega\right), \quad v > 0$$

and the domain of $\mathcal{A}(\omega)$ is given by:

$$D(\mathcal{A}(\omega)) = \left\{ \tilde{G} \in Y, \partial_v \tilde{G}, \partial_v^2 \tilde{G}, \partial_\zeta \tilde{G} \in L^2(\mathcal{D}), (21.24) \text{ holds} \right\}$$

We can then apply Proposition 21.8 which implies that $R(\mathcal{A}^*(\omega))$ is dense in $L^2(\mathcal{D})$ for any $\omega \in \mathbb{R}$ such that $\text{Ker}(\mathcal{A}(\omega)) = \{0\}$. Notice that $\mathcal{A}(\omega) \tilde{G}(\zeta, v) = 0$ if and only if $\varphi(\zeta, v) = \varphi(X) = |X|^{-(\alpha+2)} \tilde{G}(X, \omega)$ solves $\mathcal{L}(\varphi) = 0$ in $x \geq 0$, $v \in \mathbb{R}$ and $\varphi(0, rv) = \varphi(0, -v)$, $v > 0$, as well as (21.10). Since $b \neq 0$, Proposition 21.4 then implies that $\tilde{G}(\zeta, v) = 0$, whence $R(\mathcal{A}^*(\omega))$ is dense in $L^2(\mathcal{D})$. Now let

$$\tilde{\mathcal{A}}^*(\omega) = |X|^{2-i\omega} \mathcal{A}^*(\omega) = \frac{1}{|X|^{\alpha+i\omega}} \mathcal{B}^* \left(|X|^{\alpha+i\omega} G(X, \omega) \right)$$

Notice that $\tilde{\mathcal{A}}^* G(\mu X) = \tilde{\mathcal{A}}^* G(X)$. Then $R(\tilde{\mathcal{A}}^*(\omega))$ is also dense in $L^2(\mathcal{D})$. Therefore, for any $\varepsilon > 0$ and any function $\eta \in C^\infty(\mathbb{R} : L^2(\mathcal{D}))$, supported in $|\omega| \leq M$, we can find a family of functions $G(X, \omega)$, $\omega \in \mathbb{R}$, such that $\left\| \tilde{\mathcal{A}}^*(\omega) G(\cdot, \omega) - \eta(\cdot, \omega) \right\|_{L^2(\mathcal{D})} < \varepsilon$ for any $|\omega| \leq M$. Therefore, Lemma 21.5 implies that the functions $\xi(X)$ in (21.20) can be chosen in a dense set in $L^1\left(\mathcal{U}; \frac{dX}{|X|^2}\right)$. Therefore (21.21) implies that $g(X) = 0$, whence $\varphi = 0$ and the result follows. \square

Proof of Theorem 21.2. We consider first the cases (i) and (iii). The result in this case follows from Proposition 21.3.

In the case (ii) we suppose first that ψ is a bounded function satisfying $(D_v^2 - D_x)\psi = 1$ in $(x, v) \in \mathbb{R}^2$. Therefore, using standard parabolic theory we can write:

$$\psi(x, v) = -x + w(x, v) \quad \text{for } x \geq 0$$

where $(D_v^2 - D_x)w = 0$ and $w(0, v) = \psi(0, v)$, $\psi(0, \cdot)$ bounded. The classical theory of the heat equation establishes that w is uniquely determined and $|w(x, v)| \leq \sup_{v \in \mathbb{R}} |\psi(0, v)|$. But $\lim_{x \rightarrow \infty} \psi(x, v) = -\infty$, and this contradicts the boundedness of ψ . The case in which $\bar{\mathcal{L}} = D_v^2 + vD_x$ can be studied similarly. Suppose then that $\bar{\mathcal{L}} = D_v^2 + vD_x$. In this case we use the fact that ψ is a bounded solution of the Kolmogorov equation:

$$\psi_t = \psi_{vv} + v\psi_x - 1, \quad t \in \mathbb{R}, (x, v) \in \mathbb{R}^2$$

Therefore, we have the following representation formula:

$$(21.25) \quad \psi(t, x, v) = -t + \int_{\mathbb{R}^2} G(t, x - \xi, v - \eta) \psi(0, \xi, \eta) d\xi d\eta \quad \text{for } t > 0$$

where G is the fundamental solution for the Kolmogorov equation (cf. [37]). Then, the integral term in (21.25) is globally bounded, whence $\psi(t, x, v)$ is unbounded for $t \rightarrow \infty$. The contradiction concludes the Proof of (ii).

In order to prove (iv) we argue as follows. By assumption ψ is bounded. Let $m = \inf(\psi)$, $M = \sup(\psi)$. If either of them is reached as an interior point we would obtain that ψ is constant by the strong maximum principle (cf. Proposition 20.7). Let $L = \limsup_{|(x,v)| \rightarrow \infty} \psi(x, v)$ and $\liminf_{(x,v) \rightarrow (0,0)} \psi(x, v) = \kappa_0$. By assumption $\kappa_0, L \in [m, M]$. We can then, arguing by comparison, use as barrier function $\bar{\psi} = \kappa_0 + \sigma F_\beta$ in the admissible domain $\mathcal{R}_R \setminus \mathcal{R}_\varepsilon$ with R large and ε small, with $\sigma \rightarrow 0$ as $R \rightarrow \infty$, due to the fact that ψ is bounded. It then follows that $\kappa_0 - \delta \leq \psi \leq \kappa_0 + C\sigma$ in bounded sets of \mathcal{U} with $\delta > 0$ arbitrarily small. Taking the limit $R \rightarrow \infty$, $\sigma \rightarrow 0$, $\delta \rightarrow 0$ we obtain $\psi = \kappa_0$ whence the result follows. \square

We will need the following auxiliary result.

Lemma 21.10. *Suppose that $\lambda(R)$ is a positive continuous function such that*

$$\liminf_{R \rightarrow 0} \frac{\log(\lambda(R))}{\log(R)} = \zeta > 0, \quad \zeta < \infty \quad \text{for some } \zeta > 0.$$

Suppose that $K > 1$ is given. Then there exist sequences $\{R_n\}$, $\{\mu_n\}$ with $R_n \rightarrow 0$, $\mu_n \rightarrow \infty$ as $n \rightarrow \infty$ such that $\frac{\lambda(R)}{\lambda(R_n)} \leq K \left(\frac{R}{R_n}\right)^\zeta$ for $\frac{1}{\mu_n} \leq \frac{R}{R_n} \leq \mu_n$.

Proof. We claim the following. Suppose that we fix $L > 1$. We then claim that for any $\delta > 0$ (small), there exists $\bar{R} \leq \delta$ such that $\frac{\lambda(\rho)}{\lambda(\bar{R})} \leq K \left(\frac{\rho}{\bar{R}}\right)^\zeta$ for all $\rho \in \left[\frac{\bar{R}}{L}, L\bar{R}\right]$. The existence of the sequences $\{R_n\}$, $\{\mu_n\}$ then follows inductively by choosing $L = n = \mu_n$, $\delta = 1$ if $n = 1$ and $\delta = \frac{R_{n-1}}{2}$ if $n > 1$ and denoting the corresponding \bar{R} as R_n .

We prove the claim by contradiction. Suppose that it is false. Then, for any $L > 1$ fixed, there exists $R_0 > 0$ such that, for $\bar{R} \leq R_0$ there exists $\rho = \rho(\bar{R}) \in \left[\frac{\bar{R}}{L}, L\bar{R}\right]$ such that $\frac{\lambda(\rho(\bar{R}))}{\lambda(\bar{R})} > K \left(\frac{\rho(\bar{R})}{\bar{R}}\right)^\zeta$. We then define, for each $R_* \leq R_0$ a sequence inductively as follows. We define $R_1 = \rho(R_*)$ and then $R_{n+1} = \rho(R_n)$ for $n \geq 1$ as long as $R_n \leq R_0$. If $\rho(R_n) > R_0$ for some $n \geq 1$ the iteration finishes.

We now have three possibilities. (i) Given $R_* \leq R_0$, for the corresponding sequence $\{R_n\}$ we have $\liminf_{n \rightarrow \infty} R_n = 0$. (ii) Given $R_* \leq R_0$, for the corresponding sequence $\{R_n\}$ we have $0 < \liminf_{n \rightarrow \infty} R_n \leq R_0$. (iii) Given $R_* \leq R_0$ there exists N such that $R_N > R_0$. We define $\lambda_n = \lambda(R_n)$. We will denote the subsets of $(0, R_0]$ where each of the three possibilities takes place as A_1 , A_2 , A_3 respectively.

Suppose first that we are in the case (i), or more precisely that A_1 is nonempty. Then, for the corresponding R_* we have the sequence $\{R_n\}$ as defined above. We then have:

$$\lambda_n \geq K \lambda_{n-1} \left(\frac{\rho(R_{n-1})}{R_{n-1}}\right)^\zeta = K \lambda_{n-1} \left(\frac{R_n}{R_{n-1}}\right)^\zeta \geq \dots \geq K^n \lambda(R_*) \left(\frac{R_n}{R_*}\right)^\zeta$$

Then

$$\begin{aligned} \liminf_{R \rightarrow 0} \frac{\log(\lambda(R))}{\log(R)} &\leq \liminf_{n \rightarrow \infty} \frac{\log(\lambda_n)}{\log(R_n)} \\ &\leq \liminf_{n \rightarrow \infty} \frac{n \log(K) + \zeta \log(R_n) + \log(\lambda(R_*)) - \zeta \log(R_*)}{\log(R_n)} \\ &= \liminf_{n \rightarrow \infty} \frac{n \log(K) + \zeta \log(R_n)}{\log(R_n)} \\ &= \zeta + \log(K) \liminf_{n \rightarrow \infty} \left(\frac{n}{\log(R_n)}\right) \end{aligned}$$

Notice that $\rho(R) \geq \frac{R}{L}$. Then $R_n = \rho(R_{n-1}) \geq \frac{R_{n-1}}{L} \geq \dots \geq \frac{R_*}{L^n}$, whence $\log(R_n) \geq \log(R_*) - n \log(L) \geq -(n+1) \log(L)$ for n large enough. Then

$$|\log(R_n)| \leq (n+1) \log(L),$$

whence $\frac{1}{|\log(R_n)|} \geq \frac{1}{(n+1) \log(L)}$ and $-\frac{1}{|\log(R_n)|} \leq -\frac{1}{(n+1) \log(L)}$, i.e. $\frac{1}{\log(R_n)} \leq -\frac{1}{(n+1) \log(L)}$. This yields:

$$\begin{aligned} \liminf_{R \rightarrow 0} \frac{\log(\lambda(R))}{\log(R)} &\leq \zeta - \log(K) \liminf_{n \rightarrow \infty} \left(\frac{n}{(n+1) \log(L)}\right) \\ \liminf_{R \rightarrow 0} \frac{\log(\lambda(R))}{\log(R)} &\leq \zeta - \frac{\log(K)}{\log(L)} < \zeta \end{aligned}$$

This contradicts the fact that $\liminf_{R \rightarrow 0} \frac{\log(\lambda(R))}{\log(R)} = \zeta$ and then, it implies that the set A_1 is empty.

Suppose then that A_2 is nonempty. We also have the inequality:

$$\lambda_n \geq K^n \lambda(R_*) \left(\frac{R_n}{R_*} \right)^\zeta$$

whence:

$$\lambda(R_*) \leq \frac{\lambda_n}{(R_n)^\zeta} \frac{(R_*)^\zeta}{K^n}$$

where n is arbitrarily large. Then, since the limit $\lim_{n \rightarrow \infty} \left(\frac{\lambda_n}{(R_n)^\zeta} \right) = \ell = \ell_{R_*} > 0$ exists, we obtain

$$\lambda(R_*) \leq \ell_{R_*} \cdot (R_*)^\zeta \lim_{n \rightarrow \infty} \frac{1}{K^n} = 0$$

Therefore $\lambda(R_*) = 0$ if $R_* \in A_2$, which is a contradiction.

If $R_* \in A_3$ we can apply the same argument to obtain:

$$\lambda(R_*) \leq \frac{\lambda_N}{(R_N)^\zeta} \frac{(R_*)^\zeta}{K^N}$$

We now have the inequality $R_N = \rho(R_{N-1}) \leq LR_{N-1} \leq \dots \leq L^N R_*$ which implies $N \log(L) \geq \log\left(\frac{R_N}{R_*}\right)$. Then $K^N \geq \left(\frac{R_N}{R_*}\right)^{\frac{\log(K)}{\log(L)}}$, whence:

$$\lambda(R_*) \leq \frac{\lambda_N}{(R_N)^{\zeta + \frac{\log(K)}{\log(L)}}} (R_*)^{\zeta + \frac{\log(K)}{\log(L)}}$$

Given that $R_N \geq R_0$ we obtain:

$$\lambda(R_*) \leq C (R_*)^{\zeta + \frac{\log(K)}{\log(L)}}$$

with C independent of R_* . Since $A_1 = A_2 = \emptyset$ we obtain this estimate for $R_* \leq R_0$ arbitrary. Then:

$$\liminf_{R \rightarrow 0} \frac{\log(\lambda(R))}{\log(R)} \geq \zeta + \frac{\log(K)}{\log(L)} > \zeta$$

and this would give a contradiction. \square

The following asymptotic result is interesting for itself, and it will be used repeatedly in the following.

Theorem 21.11. *Let $0 < r < r_c$. Suppose that $h \in C(X)$, $\varphi \in C(X)$ satisfies (11.1) and that we have $\mathcal{L}\varphi = h$, with \mathcal{L} as in Definition 11.7. Then*

$$(21.26) \quad |\varphi(x, v) - \varphi(0, 0) - \mathcal{A}(\varphi) F_\beta(x, v) - h(0, 0) S(x, v)| = o\left(x^{\frac{2}{3}} + |v|^2\right)$$

as $x^{\frac{2}{3}} + |v|^2 \rightarrow 0$ for some suitable constant $\mathcal{A}(\varphi) \in \mathbb{R}$.

Proof. We construct sub and supersolutions with the form:

$$\varphi_\pm(x, v) = \varphi(0, 0) \pm \varepsilon \pm \bar{C} F_\beta(x, v) \pm A S(x, v),$$

where $A = 2|h(0, 0)|$, $\varepsilon > 0$, $\bar{C} > 0$. We can choose $0 < \delta_1 < \delta_2$ small and then \bar{C} large enough to obtain $\varphi_- \leq \varphi \leq \varphi_+$ in $\partial\mathcal{R}_{\delta_1} \cup \partial\mathcal{R}_{\delta_2}$. Proposition 20.6 yields then $\varphi_- \leq \varphi \leq \varphi_+$ in $\mathcal{R}_{\delta_2} \setminus \mathcal{R}_{\delta_1}$. Taking the limit $\delta_1 \rightarrow 0$ and then $\varepsilon \rightarrow 0$ we obtain:

$$|\varphi(x, v) - \varphi(0, 0)| \leq 2\bar{C} F_\beta(x, v) \quad \text{in } \mathcal{R}_\delta$$

for some $\delta > 0$. We can now prove that the limit $\frac{\varphi(x,v) - \varphi(0,0)}{F_\beta(x,v)}$ exists. To this end we define $C_- = \liminf_{R \rightarrow 0} \left[\inf_{\partial \mathcal{R}_R} \left(\frac{\varphi(x,v) - \varphi(0,0)}{F_\beta(x,v)} \right) \right]$ and $C_+ = \limsup_{R \rightarrow 0} \left[\sup_{\partial \mathcal{R}_R} \left(\frac{\varphi(x,v) - \varphi(0,0)}{F_\beta(x,v)} \right) \right]$. If $C_- = C_+$ the result would follow. Suppose then that $C_- < C_+$. Suppose first that there exist sequences $\{R_n\}$, $\{\tilde{R}_n\}$ and positive constants c_1, c_2 such that

$$\lim_{n \rightarrow \infty} \left[\sup_{\partial \mathcal{R}_{R_n}} \left(\frac{\varphi(x,v) - \varphi(0,0)}{F_\beta(x,v)} \right) \right] = C_+$$

and

$$\lim_{n \rightarrow \infty} \left[\inf_{\partial \mathcal{R}_{\tilde{R}_n}} \left(\frac{\varphi(x,v) - \varphi(0,0)}{F_\beta(x,v)} \right) \right] = C_- \text{ with } c_1 R_n \leq \tilde{R}_n \leq c_2 R_n.$$

Notice that we might have in particular cases $R_n = \tilde{R}_n$. We then define $\psi_n(x,v) = \frac{\varphi(R_n^3 x, R_n^3 v) - \varphi(0,0)}{(R_n)^{3\beta}}$. A standard compactness argument implies that for a suitable convergence subsequence $\psi_n \rightarrow \psi$ in compact sets which do not contain the singular point $(x,v) = (0,0)$ and where $\mathcal{L}\psi = 0$. Moreover, our assumptions in the sequences $\{R_n\}$, $\{\tilde{R}_n\}$, as well as the hypoellipticity properties of the operator \mathcal{L} imply the existence of points different from the singular point such that $\psi(x_1, v_1) = C_+ F_\beta(x_1, v_1)$, $\psi(x_2, v_2) = C_- F_\beta(x_2, v_2)$. Moreover $C_- F_\beta \leq \psi \leq C_+ F_\beta$. Since $C_- < C_+$, the strong maximum principle then implies then that $\psi \leq (C_+ - \delta) F_\beta$ for some $\delta > 0$ in a set $\partial \mathcal{R}_\rho$ for some $0 < \rho < \infty$. A comparison argument then yields $\psi \leq (C_+ - \delta) F_\beta$ in \mathcal{R}_ρ . We remark that the comparison argument is made in a family of domains $\mathcal{R}_\rho \setminus \mathcal{R}_\delta$, adding ε to the barrier functions and taking then the limit $\delta \rightarrow 0$ and $\varepsilon \rightarrow 0$. However, using the definition of ψ_n we then obtain that $\limsup_{R \rightarrow 0} \left[\sup_{\partial \mathcal{R}_R} \left(\frac{\varphi(x,v) - \varphi(0,0)}{F_\beta(x,v)} \right) \right] \leq (C_+ - \delta)$, a contradiction.

Suppose now that the solutions $\{R_n\}$, $\{\tilde{R}_n\}$ with the property stated above do not exist. This implies the existence of a sequence $\{R_n\}$ such that

$$\lim_{n \rightarrow \infty} \left[\sup_{\partial \mathcal{R}_{R_n}} \left| \frac{\varphi(x,v) - \varphi(0,0)}{F_\beta(x,v)} - C_+ \right| \right] = 0.$$

A comparison argument then yields $\varphi(x,v) - \varphi(0,0) \geq (C_- + \delta) F_\beta$, for some $\delta > 0$, whence $\lim_{n \rightarrow \infty} \left[\inf_{\partial \mathcal{R}_{\tilde{R}_n}} \left(\frac{\varphi(x,v) - \varphi(0,0)}{F_\beta(x,v)} \right) \right] \geq (C_- + \delta)$. The contradiction then implies that $C_- = C_+$. Therefore we have:

$$(21.27) \quad |\varphi(x,v) - \varphi(0,0) - \mathcal{A}(\varphi) F_\beta(x,v)| = o\left(x^\beta + |v|^{3\beta}\right)$$

as $(x,v) \rightarrow (0,0)$, where $\mathcal{A}(\varphi) = C_+$.

As a next step we need to prove that:

$$(21.28) \quad |\varphi(x,v) - \varphi(0,0) - \mathcal{A}(\varphi) F_\beta(x,v)| \leq C \left(x^{\frac{2}{3}} + |v|^2 \right)$$

for $(x,v) \in \partial \mathcal{R}_\delta$ for suitable $\delta > 0$, $C > 0$. We define the auxiliary function:

$$(21.29) \quad \lambda(R) = \sup_{x^{\frac{2}{3}} + |v|^2 = R^{\frac{2}{3}}} |\varphi(x,v) - \varphi(0,0) - \mathcal{A}(\varphi) F_\beta(x,v)|$$

Notice that (21.27) yields $\lambda(R) = o(R^\beta)$ as $R \rightarrow 0$.

Suppose now that $\limsup_{R \rightarrow 0} \frac{\lambda(R)}{R^{\frac{2}{3}}} = \infty$. We then have, using also the estimate $\lambda(R) = o(R^\beta)$ as $R \rightarrow 0$, that $\liminf_{R \rightarrow 0} \frac{\log(\lambda(R))}{\log(R)} = \zeta \in [\beta, \frac{2}{3}]$. Applying Lemma 21.10 we obtain the

existence of a sequence $\{R_n\}$ such that $\frac{\lambda(R)}{\lambda(R_n)} \leq K \left(\frac{R}{R_n}\right)^\zeta$ for $R \in \left[\frac{R_n}{\mu_n}, \mu_n R_n\right]$. Rescaling the solutions and defining ψ_n as before we obtain in the limit that ψ_n converges to ψ , solution of $\mathcal{L}\psi = 0$ with $|\psi| \leq K \left(x^\zeta + |v|^{3\zeta}\right)$ and $\psi \neq 0$. We now apply Theorem 21.2, which implies that $\psi = 0$. The contradiction implies that $\limsup_{R \rightarrow 0} \frac{\lambda(R)}{R^{\frac{2}{3}}} < \infty$. We then rescale the solutions again, in order to obtain a sequence of functions ψ_n of order one. The limit of this sequence satisfies $\mathcal{L}\psi = h(0,0)$ as well as $|\psi(x, v)| \leq C \left(x^{\frac{2}{3}} + |v|^2\right)$ and the boundary conditions $\psi(0, rv) = \psi(0, -v)$ for $v > 0$. After subtracting from ψ the function $h(0,0)S$ we obtain a new function $R = \psi - h(0,0)S$ satisfying $\mathcal{L}R = 0$ and $|\psi(x, v)| \leq C \left(x^{\frac{2}{3}} + |v|^2\right)$ as well as similar boundary conditions. Using Theorem 21.2 we obtain $R = 0$, whence the asymptotics (21.26) follows. \square

We have a similar result in the supercritical case.

Theorem 21.12. *Let $r > r_c$. Suppose that $h \in C(X)$, $\varphi \in C(X)$ satisfies (11.1) and that we have $\mathcal{L}\varphi = h$, with \mathcal{L} as in Definition 11.7. Then*

$$(21.30) \quad |\varphi(x, v) - \varphi(0, 0) - h(0, 0)S(x, v)| = o\left(x^{\frac{2}{3}} + |v|^2\right)$$

as $x^{\frac{2}{3}} + |v|^2 \rightarrow 0$ for some suitable constant $\mathcal{A}(\varphi) \in \mathbb{R}$.

Proof. The proof is similar to the previous Theorem. We first prove, using the corresponding Liouville's Theorem for this case, that $\lambda(R) \leq CR^{\frac{2}{3}}$ where in this case we define $\lambda(R) = \sup_{x^{\frac{2}{3}} + |v|^2 = R^{\frac{2}{3}}} |\varphi(x, v) - \varphi(0, 0)|$. We then obtain, using similar arguments, that the asymptotics is given by the function S . \square

We conclude this Section with a technical result, which provides some uniformicity in the asymptotics of the solutions near the singular point. Suppose that $\xi = \xi(x, v)$ is a C^∞ cutoff function, supported in the region $x + |v|^3 \leq \max\left\{\frac{2}{r}, 2\right\}$, and satisfying $\xi(x, v) = 1$ for $x + |v|^3 \leq 1$ as well as $\xi(0, -v) = \xi(0, rv)$, for $v > 0$, $\xi_x(x, v) = 0$ for $x \leq \frac{1}{4}$. Let

$$(21.31) \quad \mathcal{M} = \{\psi : \psi\xi \in C(X) \mid \mathcal{L}(\psi\xi) \in C(X), |\mathcal{L}(\psi\xi)| \leq 1, \\ |\psi(x, v)| \leq F_\beta(x, v) \text{ in } |x| + |v|^3 \leq 1, \psi(x, v) = o\left(x^\beta + |v|^{3\beta}\right) \text{ as } |x| + |v|^3 \rightarrow 0,$$

where F_β , β are as in Lemma 10.1. Notice that since $r < r_c$ we have $\beta > 0$. The cutoff ξ has been introduced because in later arguments we will need to use unbounded functions ψ which do not belong to $C(X)$.

Then we derive the following estimate which shows that the rate of decay of $\psi(x, v)$ as $(x, v) \rightarrow (0, 0)$ is uniform for the class of functions \mathcal{M} .

Proposition 21.13. *Let*

$$(21.32) \quad \mu(R) = \sup_{\psi \in \mathcal{M}} \sup_{x^\beta + |v|^{3\beta} = R^\beta} |\psi(x, v)|.$$

with \mathcal{M} as in (21.31). Then we have

$$\lim_{R \rightarrow 0} \frac{\mu(R)}{R^\beta} = 0.$$

Proof. We prove the result by contradiction. Suppose the opposite holds, that is, there exists $R_m \rightarrow 0$ such that $\frac{\mu(R_m)}{R_m^\beta} \geq \eta > 0$. Then there exist sequences $\{\psi_m\} \subset \mathcal{M}$, $\{x_m\}$ with

$x_m > 0$, $\{v_m\} \subset \mathbb{R}$ with $|x_m|^\beta + |v_m|^{3\beta} = R_m^\beta$ such that

$$|\psi_m| \geq \eta \left(|x_m|^\beta + |v_m|^{3\beta} \right).$$

Define

$$\Phi_m(x, v) = \frac{\psi_m \left(R_m x, R_m^{1/3} v \right)}{R_m^\beta}.$$

Then Φ_m satisfies the following.

(i)

$$(21.33) \quad |\mathcal{L}\Phi_m| \leq R_m^{2/3-\beta} \rightarrow 0 \text{ as } m \rightarrow \infty, \text{ since } \beta < 2/3.$$

(ii)

$$(21.34) \quad \Phi_m \text{ satisfies (11.1) for any } v > 0$$

(iii)

$$(21.35) \quad \Phi_m(x, v) = o \left(|x|^\beta + |v|^{3\beta} \right), \quad |x| + |v|^3 \rightarrow 0.$$

(iv)

$$(21.36) \quad |\Phi_m(x_m^*, v_m^*)| \geq \eta > 0,$$

where $(x_m^*, v_m^*) = \left(\frac{x_m}{R_m}, \frac{v_m}{R_m^{1/3}} \right)$ with $|x_m^*|^\beta + |v_m^*|^{3\beta} = 1$.

(v)

$$(21.37) \quad |\Phi_m(x, v)| \leq F_\beta(x, v).$$

By the regularizing effect of the operator \mathcal{L} and by (i), Φ_m is uniformly bounded in $W_{loc}^{1,p}(\mathbb{R}^+ \times \mathbb{R})$ for all $1 \leq p < \infty$ (cf. Theorem 12.4). Then by the Sobolev embedding, Φ_m converges to Φ^* uniformly in m on compact sets (up to subsequences) and Φ^* satisfies the following.

$$\mathcal{L}\Phi^* = 0 \text{ in } \mathcal{U}, \quad \Phi^* \text{ satisfies (11.1),}$$

$$|\Phi^*(x, v)| \leq F_\beta(x, v), \quad |\Phi^*(x^*, v^*)| \geq \eta > 0,$$

where $(x_m^*, v_m^*) \rightarrow (x^*, v^*)$ with $|x^*|^\beta + |v^*|^{3\beta} = 1$.

Let

$$C^* = \inf \left\{ C \mid \Phi^*(x, v) \leq CF_\beta(x, v) \text{ for all } (x, v) \in \mathbb{R}^+ \times \mathbb{R} \right\}.$$

Notice that $\sup_{(x,v)} \frac{\Phi^*(x,v)}{F_\beta(x,v)} = C^*$. Indeed, if $\sup_{(x,v)} \frac{\Phi^*(x,v)}{F_\beta(x,v)} > C^*$, we would obtain that, for any $\delta > 0$ sufficiently small there exists $(x_\delta, v_\delta) \in \mathbb{R}^+ \times \mathbb{R}$ such that $\Phi^*(x_\delta, v_\delta) > (C^* + \delta) F_\beta(x, v)$, whence $\sup_{(x,v)} \frac{\Phi^*(x,v)}{F_\beta(x,v)} \geq C^* + \delta$, and this contradicts the definition of C^* . If $\sup_{(x,v)} \frac{\Phi^*(x,v)}{F_\beta(x,v)} < C^*$ we would have $\sup_{(x,v)} \frac{\Phi^*(x,v)}{F_\beta(x,v)} < C^* - \delta$ for some $\delta > 0$ small, and therefore we would have $C^* \leq C^* - \delta$ which gives a contradiction too.

We now claim that:

$$(21.38) \quad \Phi^*(x, v) = C^* F_\beta(x, v)$$

To prove this we examine the different possible cases depending on the values of (x, v) where $\sup_{(x,v)} \frac{\Phi^*(x,v)}{F_\beta(x,v)}$ is achieved. At least one of the following alternatives holds:

Case (a) There exists $(\bar{x}, \bar{v}) \neq (0, 0)$ with $|(\bar{x}, \bar{v})| < \infty$ such that $\Phi^*(\bar{x}, \bar{v}) = C^* F_\beta(\bar{x}, \bar{v})$.

Case (b)

$$(21.39) \quad \limsup_{R \rightarrow 0} \sup_{|x|^\beta + |v|^{3\beta} = R^\beta} \frac{\Phi^*(x, v)}{F_\beta(x, v)} = C^*.$$

Case (c)

$$\limsup_{R \rightarrow \infty} \sup_{|x|^\beta + |v|^{3\beta} = R^\beta} \frac{\Phi^*(x, v)}{F_\beta(x, v)} = C^*.$$

In the case (a) we notice that $\Phi^*(x, v) \leq C^* F_\beta(x, v)$ for all $(x, v) \in \mathbb{R}^+ \times \mathbb{R}$. We then obtain a contradiction using the Strong Maximum Principle (cf. Proposition 20.7) if $\Phi^*(x, v) \neq C^* F_\beta(x, v)$, since we would obtain a minimum of the function $(C^* F_\beta(x, v) - \Phi^*)$ which satisfies $\mathcal{L}(C^* F_\beta(x, v) - \Phi^*) = 0$.

In the case (b), suppose $\Phi^*(x, v) \neq C^* F_\beta(x, v)$, then we may assume without loss of generality that $\Phi^*(x, v) < C^* F_\beta(x, v)$, for $|x|^\beta + |v|^{3\beta} = 1$. Then, using Proposition 20.6 as in the proof of Theorem 21.11, i.e. making the comparison in a region removing a small set near the singular point, adding a small value $\varepsilon > 0$ and taking the limit as the domain converges to the whole $\mathbb{R}^+ \times \mathbb{R}$ we can show that $\Phi^*(x, v) \leq (C^* - \delta) F_\beta(x, v) + \varepsilon$, for some $\delta > 0$ small and for arbitrary $\varepsilon > 0$ small. By letting $\varepsilon \rightarrow 0$, we get $\Phi^*(x, v) \leq (C^* - \delta) F_\beta(x, v)$, which contradicts to (21.39).

In the case (c), there exists a sequence (\bar{x}_l, \bar{v}_l) such that $|(\bar{x}_l, \bar{v}_l)| \rightarrow \infty$ and

$$\lim_{l \rightarrow \infty} \frac{\Phi^*(\bar{x}_l, \bar{v}_l)}{F_\beta(\bar{x}_l, \bar{v}_l)} = C^*.$$

Define

$$\Phi_l^{**}(x, v) = \frac{\Phi^*(\rho_l x, \rho_l^{1/3} v)}{\rho_l^\beta}, \quad \rho_l^\beta = |\bar{x}_l|^\beta + |\bar{v}_l|^{3\beta}.$$

Then there exist $\Phi_\infty^{**}(x, v) \in \mathcal{M}$ and $(\bar{x}_\infty, \bar{v}_\infty) \in \mathbb{R}_+ \times \mathbb{R}$ such that $(\frac{\bar{x}_l}{\rho_l}, \frac{\bar{v}_l}{\rho_l^{1/3}}) \rightarrow (\bar{x}_\infty, \bar{v}_\infty)$ with $|\bar{x}_\infty|^\beta + |\bar{v}_\infty|^{3\beta} = 1$ and $\Phi_l^{**}(x, v) \rightarrow \Phi_\infty^{**}(x, v)$, as $l \rightarrow \infty$, $\mathcal{L}(\Phi_\infty^{**}) = 0$, $\Phi_\infty^{**}(x, v) \leq C^* F_\beta(x, v)$ and

$$\Phi_\infty^{**}(\bar{x}_\infty, \bar{v}_\infty) = C^* F_\beta(\bar{x}_\infty, \bar{v}_\infty).$$

Then by Proposition 20.7 we have

$$\Phi_\infty^{**}(x, v) = C^* F_\beta(x, v).$$

Thus for $\delta > 0$ given small and l large and for $1/4 \leq |x|^\beta + |v|^{3\beta} \leq 4$,

$$\Phi_l^{**}(x, v) \geq \left(C^* - \frac{\delta}{10^6}\right) F_\beta(x, v),$$

which in turn implies, for $\rho_l/4 \leq |x|^\beta + |v|^{3\beta} \leq 4\rho_l$,

$$\Phi^*(x, v) \geq \left(C^* - \frac{\delta}{10^6}\right) F_\beta(x, v).$$

Now suppose $\Phi^*(x, v) \neq C^* F_\beta(x, v)$, then, using the comparison argument above it would follow that $\Phi^*(x, v) \leq (C^* - \delta) F_\beta(x, v)$ for $\rho_l = |x|^\beta + |v|^{3\beta}$, which is a contradiction.

Thus the claim (21.38) is proved. Next we claim $C^* = 0$. Suppose $C^* \neq 0$. Take $C^* > 0$ without loss of generality. Then for any given $\delta > 0$ small, for $|x|^\beta + |v|^{3\beta} = 1$, $\Phi_m \geq (C^* - \delta) F_\beta$ for large m since $\Phi_m \rightarrow \Phi^*$ as $m \rightarrow \infty$ and $\Phi^*(x, v) = C^* F_\beta(x, v)$. We recall a super-solution $S(x, v)$ with $S \sim |x|^{2/3} + |v|^2$ near $(0, 0)$ and $\mathcal{L}S = 1$ as in Lemma 21.1.

Then we construct a sub-solution of the form

$$Z(x, v) = (C^* - 2\delta) F_\beta(x, v) - R_m^{2/3-\beta} S(x, v),$$

so that we have by comparison principle,

$$\Phi_m(x, v) \geq Z(x, v), \quad \text{for } |x|^\beta + |v|^{3\beta} \leq 1.$$

Therefore, we have

$$\Phi_m(x, v) \geq (C^* - 3\delta) F_\beta(x, v), \text{ for } |x|^\beta + |v|^{3\beta} \leq 1,$$

which contradicts to (21.35). Thus the claim is proved, that is, $C^* = 0$. Then $\Phi^*(x, v) = 0$, which contradicts to (21.36). Thus the proof is complete. \square

22. THE OPERATORS Ω_σ ARE MARKOV PREGENERATORS.

We will denote in the following as Ω_σ any of the operators $\Omega_{t,sub}$, $\Omega_{nt,sub}$, $\Omega_{pt,sub}$, Ω_{sup} defined in Sections 13.1, 13.2, 13.3, 13.4. In the rest of this Section we check that the four operators Ω_σ defined above are Markov generators in the sense of Definition 18.5, and therefore that they define Markov generators due to Hille-Yosida Theorem (cf. Theorem 18.7). We first check that the operators Ω_σ are Markov pregenerators.

Proposition 22.1. *The operators $\Omega_{t,sub}$, $\Omega_{nt,sub}$, $\Omega_{pt,sub}$, Ω_{sup} defined in Sections 13.1, 13.2, 13.3, 13.4 are Markov pregenerators in the sense of Definition 18.1.*

We will prove Proposition 22.1 with the help of several Lemmas.

Lemma 22.2. *Suppose that $\Omega_\sigma \in \{\Omega_{t,sub}, \Omega_{nt,sub}, \Omega_{pt,sub}, \Omega_{sup}\}$. The domains $\mathcal{D}(\Omega_\sigma)$ defined in (13.2), (13.4), (13.6), (13.8) are dense in $C(X)$ endowed with the uniform topology.*

Proof. We need to check that the domains $\mathcal{D}(\Omega_\sigma)$ are dense in $C(X)$ with the uniform norm $\|\cdot\|$ in (11.4). To this end we argue as follows. Let $\xi \in C(X)$, then we want to find an approximate sequence in $\mathcal{D}(\Omega)$ converging uniformly to ξ in X . To this end, for given $\rho > 0$, first we introduce the following function $\lambda \in C^\infty[0, \infty)$ so that

$$0 \leq \lambda(s) \leq 1; \quad \lambda(s) = 1 \text{ for } 0 \leq s \leq \rho; \quad \lambda(s) = 0 \text{ for } s \geq 2\rho$$

and we define $\eta(x, v)$ by

$$(22.1) \quad \eta(x, v) = \begin{cases} \lambda(x + |v|^3), & v < 0 \\ \lambda(x + |\frac{v}{r}|^3), & v \geq 0. \end{cases}$$

Note that η is continuous in X , $\eta(0, -v) = \eta(0, rv)$ for $v > 0$, and $\psi = \mathcal{A}(\eta) = \mathcal{L}(\eta) = 0$ in (13.1) since η is constant near the origin. Let $\varepsilon > 0$ be given. Choose $\tilde{\xi} = \eta\xi(0, 0) + (1 - \eta)\xi$, where η is given in (22.1). Then $\eta = 1$ on $\{x + |v|^3 \leq \rho \text{ and } v < 0, \text{ or } x + |\frac{v}{r}|^3 \leq \rho \text{ and } v \geq 0\}$ and $\eta = 0$ for $\{x + |v|^3 \geq 2\rho \text{ and } v < 0, \text{ or } x + |\frac{v}{r}|^3 \geq 2\rho \text{ and } v \geq 0\}$ and $0 \leq \eta \leq 1$ otherwise. Then it is easy to see, since ξ is continuous up to the boundary,

$$\|\xi - \tilde{\xi}\| = \|\eta(\xi - \xi(0, 0))\| \leq \varepsilon/2,$$

for sufficiently small ρ . Write

$$\tilde{\xi} = \xi(0, 0) + (1 - \eta)(\xi - \xi(0, 0)) =: \xi_1 + \xi_2.$$

Then $\xi_1 \in \mathcal{D}(\Omega)$ since it is a constant function. We now consider $\xi_2 = (1 - \eta)(\xi - \xi(0, 0))$ and will find an approximation in $\mathcal{D}(\Omega)$ to it. Note that ξ_2 is 0 near the origin. We may assume that ξ_2 is compactly supported. Let us consider the standard mollifiers $\{\zeta^\delta\}_{\delta>0}$. By extending ξ_2 to zero outside of X , we consider the following approximation of ξ_2 via convolution

$$\xi_2^\delta(x, v) = \int_{|x-y|^2 + |v-w|^2 \leq \delta} \zeta^\delta(x - y, v - w) \xi_2(y, w) dy dw$$

Then for sufficiently small $\delta (< r\rho/2)$, we obtain $\xi_2^\delta = 0$ in a small neighborhood of the origin and that $\|\xi_2^\delta - \xi_2\| \leq \varepsilon/2$. However, this ξ_2^δ does not belong to $\mathcal{D}(\Omega)$ in general, since ξ_2^δ and $\mathcal{L}\xi_2^\delta$ do not satisfy the boundary condition (11.1). In order to find the approximation of ξ_2 that lies in $\mathcal{D}(\Omega)$, we will construct a smooth function φ^δ in a thin strip $Y = \{0 < x < \delta/2, v < -\delta/2\}$

near the lower boundary but away from the singular set so that it is close to ξ_2 in that strip Y and moreover, $\varphi^\delta(0, v) = \xi_2^\delta(0, -rv)$ and $\mathcal{L}\varphi^\delta(0, v) = \mathcal{L}\xi_2^\delta(0, -rv)$. This will be achieved by solving the parabolic equation in Y :

$$\begin{aligned} \partial_x \varphi^\delta(x, v) &= -\frac{1}{v} \partial_v^2 \varphi^\delta - r \partial_x \xi_2^\delta(x, -rv) + \frac{1}{v} \partial_v^2 \xi_2^\delta(x, -rv) \quad \text{for } 0 < x < \delta/2, \quad v < -\delta/2 \\ \varphi^\delta(0, v) &= \xi_2^\delta(0, -rv), \quad \varphi^\delta(x, -\frac{\delta}{2}) = 0 \quad \text{for } 0 < x < \delta/2. \end{aligned}$$

Here we view x as a time-like variable. Since $-v$ is away from zero, the existence and regularity of the solution follow from classical parabolic theory. Note that the parabolic equation guarantees $\mathcal{L}\varphi^\delta(0, v) = \mathcal{L}\xi_2^\delta(0, -rv)$ and the initial data gives $\varphi^\delta(0, v) = \xi_2^\delta(0, -rv)$. In addition, since $\|\xi_2^\delta - \xi_2\| \leq \varepsilon/2$ and $\varphi^\delta(0, v) = \xi_2^\delta(0, -rv) \sim \xi_2(0, v)$, and by continuity of ξ_2 , we deduce that $\sup_Y |\varphi^\delta - \xi_2| \leq \varepsilon/2$ for sufficiently small δ .

By introducing another smooth cutoff function χ so that $\chi(x, v) = 1$ in $Y' = \{0 < x < \delta/4, \quad v < -3\delta/4\}$, $\chi(x, v) = 0$ in $X \setminus Y$, and $0 \leq \chi \leq 1$, we now let $\tilde{\xi}_2^\delta := \chi \varphi^\delta + (1 - \chi) \xi_2^\delta$. Then by construction, we deduce that $\tilde{\xi}_2 \in \mathcal{D}(\Omega)$ and moreover,

$$\|\xi_2 - \tilde{\xi}_2^\delta\| \leq \varepsilon/2,$$

for sufficiently small $\delta > 0$. Therefore,

$$\|\xi - (\xi_1 + \tilde{\xi}_2^\delta)\| = \|\xi - \tilde{\xi} + \xi_2 - \tilde{\xi}_2^\delta\| \leq \varepsilon.$$

Since ε is arbitrary, we are done. \square

We now prove the following:

Lemma 22.3. *Suppose that $\Omega_\sigma \in \{\Omega_{t,sub}, \Omega_{nt,sub}, \Omega_{pt,sub}, \Omega_{sup}\}$. Let $g \in C(X)$ and suppose that there exists $\varphi \in \mathcal{D}(\Omega_\sigma)$ such that $\varphi - \lambda \Omega_\sigma \varphi = g$ for some $\lambda \geq 0$. Then $\min_{\zeta \in X} \varphi(\zeta) \geq \min_{\zeta \in X} g(\zeta)$.*

Proof. In order to prove the result we consider several (not mutually exclusive) subcases, namely:

- (a) $\varphi(x_0, v_0) = \min_X \varphi$, with $x_0 > 0, \quad v_0 \in \mathbb{R}$.
- (b) $\varphi(0, v_0) = \min_X \varphi$, with $v_0 \in \mathbb{R}$.
- (c) $\varphi(\infty) = \min_X \varphi$.
- (d) $\varphi(0, 0) = \min_X \varphi$.

By assumption, in the case (a) we have that $\varphi(x, v) - \mathcal{L}\varphi(x, v) = g(x, v)$ for (x, v) in a neighbourhood of (x_0, v_0) contained in \mathcal{U} . We now claim that $\mathcal{L}\varphi(x_0, v_0) \geq 0$. Suppose the opposite, i.e. $\mathcal{L}\varphi(x_0, v_0) < 0$. Then, since $g, \varphi \in C(X)$ there exists an admissible domain Ξ (cf. Definition 12.1), such that $(x_0, v_0) \in \Xi$ and $\mathcal{L}\varphi(x, v) \leq -\kappa$ for $(x, v) \in \Xi$ with $\kappa > 0$. Moreover, we can assume also that $\text{dist}((x_0, v_0), \partial_a \Xi) \geq \frac{d(\Xi)}{4}$ with $d(\Xi)$ as in (20.7). Due to (a) we have $\varphi(x_0, v_0) \leq \min_{\partial_a \Xi} \varphi$. Applying Proposition 20.6 to $\bar{\varphi} = \varphi - \varphi(x_0, v_0)$ it then follows that $0 = \bar{\varphi}(x_0, v_0) \geq C_* \kappa > 0$, a contradiction. Therefore $\mathcal{L}\varphi(x_0, v_0) \geq 0$ and since $\Omega_\sigma \varphi(x_0, v_0) = \mathcal{L}\varphi(x_0, v_0)$ the result stated in the Lemma follows in this case.

In the case (b) we argue similarly. We just take a domain Ξ with the form $(0, x_2) \times (v_1, v_2)$ such that $(x_0, v_0) \in \partial \Xi$, and $\text{dist}((x_0, v_0), \partial_a \Xi) \geq \frac{d(\Xi)}{4}$. Notice that we use the identification of the points in the definition of X (cf. (11.1)) and that the points of the domain Ξ are to the left of (x_0, v_0) in the sense of Definition 11.2. If $\mathcal{L}\varphi(x_0, v_0) < 0$ we obtain a contradiction as in the case (a) using Proposition 20.6.

In the case (c), using the fact that $\mathcal{L}\varphi$ is continuous in X it follows that, for any given any $\varepsilon^* > 0$ we can choose R such that if $|(x, v)| \geq R$ we have $\varphi(x, v) \leq \min_X \varphi + \varepsilon^*$. Suppose that $\mathcal{L}\varphi(\infty) < 0$, we claim that this implies the existence of (\bar{x}, \bar{v}) with $|(\bar{x}, \bar{v})| \geq R$ such that

$\varphi(\bar{x}, \bar{v}) > \min_X \varphi + C_0$, with $C_0 > 0$ independent of ε^* . To prove this we choose $(x_0, v_0) = (x_0, 1)$ with x_0 large and we define an admissible domain $\Xi = (x_0 - 1, x_0 + 1) \times (-2, 2)$. By assumption $\mathcal{L}\varphi(x, v) \leq -\kappa$ for some $\kappa > 0$ independent of ε^* and $(x, v) \in \Xi$. Suppose that $\varphi(x, v) \leq \min_X \varphi + \varepsilon^*$ for any $(x, v) \in \partial_a \Xi$. Then, a comparison argument, as in the previous cases, yields $\varphi(x_0, 1) \leq \min_X \varphi + \varepsilon^* - C_1 \kappa$ for some constant $C_1 > 0$ independent of ε^* . Notice that the independence of C_1 follows from the invariance of the operator \mathcal{L} under translations in x . Since ε^* can be chosen arbitrarily small it then follows that $\varphi(x_0, 1) \leq \min_X \varphi - \frac{C\kappa}{2}$ if x_0 is large enough. However, since $\varphi \in C(X)$ this implies $\varphi(\infty) \leq \min_X \varphi - \frac{C\kappa}{2}$, but this contradicts the fact that $\min_X \varphi = \varphi(\infty)$. Therefore $\mathcal{L}\varphi(\infty) \geq 0$ whence $\varphi(\infty) \geq \varphi(\infty) - \mathcal{L}\varphi(\infty) = g(\infty)$ and the result follows in this case.

Suppose now that (d) holds. Notice that in the case of the four operators

$$\Omega_{t,sub}, \Omega_{nt,sub}, \Omega_{pt,sub}, \Omega_{sup}$$

we have that $\Omega_\sigma(0, 0) = (\mathcal{L}\varphi)(0, 0)$. Moreover, all the functions in the domain of the four operators satisfy (13.1).

In the case of the operator $\Omega_{t,sub}$ we have, by definition $(\Omega_{t,sub}\varphi)(0, 0) = 0$, whence the result immediately follows. In order to show this property for $\Omega_{pt,sub}$, notice that the asymptotics (13.1), combined with the fact that $F_\beta(x, v) > 0$ (cf. Proposition 10.2) implies that, since $\varphi(0, 0) = \min_X \varphi$, we have $\mathcal{A}(\varphi) \geq 0$. Therefore, using that $(\Omega_{pt,sub}\varphi)(0, 0) = \mu_* |C_*| \mathcal{A}(\varphi)$, with $\mu_* > 0$, we obtain that (iii) also holds for the operator $\Omega_{pt,sub}$. In the case of the operators $\Omega_{nt,sub}$ we have, due to the definition of its domain, that $\mathcal{A}(\varphi) = 0$. We now claim that this implies that $(\mathcal{L}\varphi)(0, 0) = 0$. Suppose that $(\mathcal{L}\varphi)(0, 0) = \kappa \neq 0$. Theorem 21.11, combined with the boundary condition $\mathcal{A}(\varphi) = 0$ yields the asymptotics:

$$(22.2) \quad \varphi(x, v) - \varphi(0, 0) - \kappa S(x, v) = o\left(x^{\frac{2}{3}} + |v|^2\right)$$

as $(x, v) \rightarrow (0, 0)$. Lemma 21.1 implies that $S(x, v)$ takes positive and negative values in any neighbourhood of $(x, v) = (0, 0)$. Therefore, if $\kappa \neq 0$ we cannot have $\varphi(0, 0) = \min_X \varphi$. This rules out the possibility (d) in the case of the operator $\Omega_{nt,sub}$.

It only remains to consider the operator Ω_{sup} . The asymptotics of $\varphi(x, v)$ near the singular point can be computed in this case using Theorem 21.12. We obtain then the asymptotics (22.2). Since S changes sign in any neighbourhood of the singular point we obtain that $\varphi(0, 0) \neq \min_X \varphi$ if $\kappa \neq 0$. \square

Proof of Proposition 22.1. Notice that for all the operators

$$\Omega_\sigma \in \{\Omega_{t,sub}, \Omega_{nt,sub}, \Omega_{pt,sub}, \Omega_{sup}\}$$

we have that $1 \in \mathcal{D}(\Omega_\sigma)$ and $\Omega_\sigma 1 = 0$ (cf. (13.3), (13.5), (13.7), (13.9)). Therefore (i) in Definition 18.1 follows. The property (ii) in Definition 18.1 follows from Lemma 22.2 and the property (iii) is a consequence of Lemma 22.3. \square

23. THE OPERATORS Ω_σ ARE MARKOV GENERATORS.

In this Section we prove that the operators Ω_σ defined in Sections 13.1, 13.2, 13.3, 13.4 are Markov generators in the sense of Definition 18.5. Due to Proposition 22.1 we just need to prove that the operators Ω_σ are closed and that $\mathcal{R}(I - \lambda\Omega_\sigma) = C(X)$ for $\lambda > 0$ small. The proof of these results is a bit technical, due to the conditions imposed at the singular point $(x, v) = (0, 0)$ in the definition of the domains $\mathcal{D}(\Omega_\sigma)$.

23.1. Closure of Ω_σ . We first show that the operators Ω_σ are closed, that is, the graph $\{(\varphi, \Omega_\sigma \varphi) \mid \varphi \in \mathcal{D}(\Omega_\sigma)\}$ are closed in $C(X) \times C(X)$. First we establish the closure property away from the singular set $(0, 0)$.

Lemma 23.1. *Let Ω_σ be one of the operators $\Omega_{t,sub}$, $\Omega_{nt,sub}$, $\Omega_{pt,sub}$, Ω_{sup} . Assume $(\varphi_m, \Omega_\sigma \varphi_m)$ with $\varphi_m \in \mathcal{D}(\Omega_\sigma)$ converges to (φ, w) in $C(X) \times C(X)$. Then*

$$\varphi, D_x \varphi, D_v \varphi, D_v^2 \varphi \in L_{loc}^p(\mathcal{U})$$

for any $p \in (1, \infty)$ and $w = (D_v^2 + v D_x) \varphi$ in \mathcal{U} .

Proof. Since $w_n = \mathcal{L}(\varphi_n) \rightarrow w \in C(X)$, we have the following, for any $\zeta \in \mathcal{F}(\mathcal{U})$:

$$\begin{aligned} \int \varphi \mathcal{L}^*(\zeta) &\leftarrow \int \varphi_n \mathcal{L}^*(\zeta) = \int \mathcal{L}(\varphi_n) \zeta \\ &= \int w_n \zeta \rightarrow \int w \zeta. \end{aligned}$$

Thus we get

$$w = \mathcal{L}(\varphi) \in C(X).$$

The regularity properties of φ are a consequence of the hypoellipticity properties of the operator \mathcal{L} (cf. Theorem 12.4). \square

The uniform estimate obtained in Proposition 21.13 enables us to derive information near the singular point for sequences $\{\varphi_m\}$ such that $\{\Omega_\sigma \varphi_m\}$ is convergent in $C(X)$.

Lemma 23.2. *Let $\{(\varphi_m, \Omega_\sigma \varphi_m)\}$ be a sequence of $C(X) \times C(X)$ with $\{\varphi_m\} \subset \mathcal{D}(\Omega_\sigma)$ where Ω_σ is any of the operators $\Omega_{t,sub}$, $\Omega_{nt,sub}$, $\Omega_{pt,sub}$. Let us assume that the sequence $\{(\varphi_m, \Omega_\sigma \varphi_m)\}$ converges to (φ, w) in $C(X) \times C(X)$. Then the sequence of numbers $\{\mathcal{A}(\varphi_m)\}$ defined by means of (13.1) is a Cauchy sequence and therefore $\lim_{m \rightarrow \infty} \mathcal{A}(\varphi_m) = L$ exists.*

Proof. In the case of the operators $\Omega_{nt,sub}$ we have, due to (13.4) that $\mathcal{A}(\varphi_m) = 0$ and the result follows trivially. Suppose then that Ω_σ is one of the operators $\Omega_{t,sub}$, $\Omega_{pt,sub}$. We first show that $\{\mathcal{A}(\varphi_m)\}$ is uniformly bounded in m . Let $\varphi_n(x, v) = \varphi_n(0, 0) + \mathcal{A}(\varphi_n) F_\beta(x, v) + \psi_n(x, v)$. By assumption $\varphi_n \rightarrow \varphi$, $\mathcal{L}\varphi_n \rightarrow \mathcal{L}\varphi$ in $C(X)$. Then $|\mathcal{L}\varphi_n| \leq A$, for some $0 < A < \infty$ and $|\varphi_n(x, v) - \varphi_n(0, 0)| \leq C$. We consider a super-solution $\bar{\varphi}_+$ and a sub-solution $\bar{\varphi}_-$ of the form

$$\bar{\varphi}_\pm(x, v) = \bar{C} F_\beta(x, v) \pm A S(x, v),$$

where $S(x, v)$ satisfies $\mathcal{L}S = 1$ as in Lemma 21.1. Then $\bar{\varphi}_+(x, v) \geq \frac{\bar{C}}{2} F_\beta(x, v)$ and $\bar{\varphi}_-(x, v) \leq \frac{\bar{C}}{2} F_\beta(x, v)$ in $x + |v|^3 = \rho$ if $\rho > 0$ is sufficiently small. Then $|\varphi_n(x, v) - \varphi_n(0, 0)| \leq K \bar{\varphi}_\pm(x, v)$ for $x + |v|^3 = \rho$, with $K > 0$ sufficiently large. On the other hand since $\varphi_n \in C(X)$, given $\varepsilon > 0$ we have $|\varphi_n(x, v) - \varphi_n(0, 0)| \leq \varepsilon$ if $x + |v|^3 = \delta$ for any $\delta > 0$ sufficiently small, depending on ε . Then the Weak Maximum Principle in Proposition 20.6 applied to $\varphi_n(x, v) - \varphi_n(0, 0)$ and $(\bar{\varphi}_\pm(x, v) \pm \varepsilon)$ yields

$$|\varphi_n(x, v) - \varphi_n(0, 0)| \leq \frac{K\bar{C}}{2} F_\beta(x, v) + \varepsilon, \text{ for all } (x, v) \text{ with } \delta \leq x + |v|^3 \leq \rho$$

and taking the limit $\delta \rightarrow 0$, $\varepsilon \rightarrow 0$ we obtain:

$$|\varphi_n(x, v) - \varphi_n(0, 0)| \leq \frac{K\bar{C}}{2} F_\beta(x, v), \text{ for all } (x, v) \text{ with } 0 \leq x + |v|^3 \leq \rho.$$

This implies

$$\left| \mathcal{A}(\varphi_n) + \frac{\psi_n(x, v)}{F_\beta(x, v)} \right| \leq \frac{K\bar{C}}{2}, \text{ for all } (x, v) \text{ with } x + |v|^3 \leq \rho.$$

Then applying Proposition 21.13 yields, as $x + |v|^3 \rightarrow 0$,

$$|\mathcal{A}(\varphi_n)| \leq \frac{K\bar{C}}{2}.$$

Next, we show that $\{\mathcal{A}(\varphi_m)\}$ is a Cauchy sequence. A similar argument can be applied to $\varphi_n(x, v) - \varphi_m(x, v) - \varphi_n(0, 0) - \varphi_m(0, 0)$ for large n, m to obtain, for all (x, v) with $x + |v|^3 \leq \rho$,

$$(23.1) \quad |\varphi_n(x, v) - \varphi_m(x, v) - \varphi_n(0, 0) - \varphi_m(0, 0)| \leq \varepsilon F_\beta(x, v),$$

where the $\varepsilon > 0$ on the right-hand side of (23.1) can be chosen arbitrarily small due to the fact that $\{\varphi_n\}$ converges uniformly in X , in particular at $(x, v) = 0$ and for $x + |v|^3 = \rho > 0$. Then:

$$\left| \mathcal{A}(\varphi_n) - \mathcal{A}(\varphi_m) + \frac{\psi_n(x, v) - \psi_m(x, v)}{F_\beta(x, v)} \right| \leq \varepsilon, \text{ for all } (x, v) \text{ with } x + |v|^3 \leq \rho$$

where ψ_n, ψ_m are chosen as in (13.1) and m, n large enough. The definition of \mathcal{M} in (21.31) implies that

$$(23.2) \quad |\mathcal{A}(\varphi_n) - \mathcal{A}(\varphi_m)| \leq \varepsilon + \frac{\mu(R)}{F_\beta(x, v)}, \text{ for all } (x, v) \text{ with } x + |v|^3 = R \leq \rho$$

Notice that the right-hand side of (23.2) is independent of n, m . Using Proposition 21.13 and taking the limit $R \rightarrow \infty$ we obtain:

$$|\mathcal{A}(\varphi_n) - \mathcal{A}(\varphi_m)| \leq \varepsilon$$

if n, m are sufficiently large. Therefore $\{\mathcal{A}(\varphi_m)\}$ is a Cauchy sequence and the result follows. \square

We show in the following lemma that the limit of functions in the domain of the operators Ω_σ has the asymptotics required near the singular set.

Lemma 23.3. *Let $\{(\varphi_m, \Omega_\sigma \varphi_m)\}$ be a sequence of $C(X) \times C(X)$ with $\{\varphi_m\} \subset \mathcal{D}(\Omega_\sigma)$ where Ω_σ is any of the operators $\Omega_{t, \text{sub}}, \Omega_{nt, \text{sub}}, \Omega_{pt, \text{sub}}, \Omega_{\text{sup}}$. Suppose that the functions φ_m satisfy (13.1), i.e., they can be written as:*

$$(23.3) \quad \varphi_m(x, v) = \varphi_m(0, 0) + \mathcal{A}(\varphi_m) F_\beta(x, v) + \psi_m(x, v)$$

where ψ_m satisfies the last identity in (13.1). Let us assume that the sequence $\{(\varphi_m, \Omega_\sigma \varphi_m)\}$ converges to (φ, w) in $C(X) \times C(X)$. Then φ satisfies (13.1) for suitable $\mathcal{A}(\varphi)$ and ψ . Moreover, we have:

$$(23.4) \quad \lim_{m \rightarrow \infty} \varphi_m(0, 0) = \varphi(0, 0),$$

$$(23.5) \quad \lim_{m \rightarrow \infty} \mathcal{A}(\varphi_m) = \mathcal{A}(\varphi),$$

$$(23.6) \quad \lim_{m \rightarrow \infty} \psi_m(x, v) = \psi(x, v),$$

$$(23.7) \quad \lim_{m \rightarrow \infty} \mathcal{L}\varphi_m(x, v) = w(x, v) = \mathcal{L}\varphi(x, v)$$

$$(23.8) \quad \lim_{m \rightarrow \infty} (\mathcal{L}\psi_m)(x, v) = (\mathcal{L}\psi)(x, v),$$

Proof. We define $\psi_m(x, v)$ by means of (23.3). Using the assumptions in Lemma 23.3 as well as Lemma 23.2 we obtain that $\lim_{m \rightarrow \infty} \psi_m(x, v)$ exists, uniformly in X and:

$$(23.9) \quad \psi(x, v) := \lim_{m \rightarrow \infty} \psi_m(x, v) = \varphi(x, v) - \varphi(0, 0) - L F_\beta(x, v)$$

with L as in Lemma 23.2. Notice that, using comparison arguments as in the Proof of Lemma 23.2 we obtain the uniform estimate $|\psi_m(x, v)| \leq C F_\beta(x, v)$ for some suitable $C > 0$ independent of m and $x + |v|^3 \leq 1$. Moreover, we have $\psi_m(x, v) = o(x + |v|^3)$ as $(x, v) \rightarrow 0$.

Then $\frac{\psi_m}{C} \in \mathcal{M}$ with \mathcal{M} as in (21.31). Using Proposition 21.13 we obtain the uniform estimate $|\psi_m(x, v)| \leq \mu(R)$ if $x + |v|^3 = R$, with $\lim_{R \rightarrow 0} \frac{\mu(R)}{R^\beta} = 0$. Then $|\psi(x, v)| \leq \mu(R)$

if $x + |v|^3 = R$, whence ψ satisfies the last identity in (13.1) and therefore φ satisfies (13.1). Using then (23.9) we obtain $L = \mathcal{A}(\varphi)$. We then have (23.5), (23.6). The identity (23.4) is a consequence of the convergence of $\{\varphi_m\}$ in $C(X)$. The identity (23.7) follows from the definition of the operator \mathcal{L} in Definition 11.7 since we can take the limit on the left-hand side of that formula. Finally (23.8) follows from the fact that $\mathcal{L}F_\beta = 0$. \square

The closure of the operators Ω_σ is just a consequence of the two previous Lemmas.

Proposition 23.4. *Let $\{(\varphi_m, \Omega_\sigma \varphi_m)\}$ be a sequence of $C(X) \times C(X)$ with $\{\varphi_m\} \subset \mathcal{D}(\Omega_\sigma)$ where Ω_σ is any of the operators $\Omega_{t,sub}$, $\Omega_{nt,sub}$, $\Omega_{pt,sub}$, Ω_{sup} . Assume that $\{(\varphi_m, \Omega_\sigma \varphi_m)\}$ converges to (φ, w) in $C(X) \times C(X)$. Then $\varphi \in \mathcal{D}(\Omega_\sigma)$ and $w = \mathcal{L}\varphi$.*

Remark 23.5. *Notice that Proposition 23.4 just means that the operators Ω_σ are closed.*

Proof. It is an immediate consequence from Lemmas 23.2 and 23.3. \square

23.2. $\mathcal{R}(I - \lambda\Omega_\sigma) = C(X)$ for $\lambda > 0$. In order to conclude the Proof of the fact that the operators Ω_σ are Markov generators it only remains to prove that $\mathcal{R}(I - \lambda\Omega_\sigma) = C(X)$ for $\lambda > 0$ small (cf. Definition 18.5). This is equivalent to prove that it is possible to solve the problems

$$(23.10) \quad (\varphi - \lambda\Omega_\sigma \varphi) = g, \quad g \in C(X)$$

with $\varphi \in C(X) \cap \mathcal{D}(\Omega_\sigma)$ for any $\lambda > 0$ small, $\Omega_\sigma \in \{\Omega_{t,sub}, \Omega_{nt,sub}, \Omega_{pt,sub}, \Omega_{sup}\}$.

The definition of the operators Ω_σ in Section 13 and in particular the choice of domains $\mathcal{D}(\Omega_\sigma)$ (cf. (13.2), (13.4), (13.6), (13.8)) allows us to reformulate (23.10) by means of PDE problems with suitable boundary conditions at the singular point $(x, v) = (0, 0)$. More precisely, the equation (23.10) is equivalent in the case of the four operators $\Omega_{t,sub}$, $\Omega_{nt,sub}$, $\Omega_{pt,sub}$, Ω_{sup} to the equation:

$$(23.11) \quad (\varphi - \lambda\mathcal{L}\varphi) = g \text{ in } \mathcal{U}, \quad \varphi \in C(X), \quad \varphi(0, rv) = \varphi(0, -v), \quad v > 0$$

where φ satisfies (13.1) and we impose the following boundary conditions for each of the cases:

$$(23.12) \quad \varphi(0, 0) = g(0, 0) \text{ for } \Omega_{t,sub} \text{ (cf. (13.2))}$$

$$(23.13) \quad \mathcal{A}(\varphi) = 0 \text{ for } \Omega_{nt,sub} \text{ (cf. (13.4))}$$

$$(23.14) \quad \lambda\mu_* |C_*| \mathcal{A}(\varphi) = \lambda(\mathcal{L}\varphi)(0, 0) = (\varphi(0, 0) - g(0, 0)) \text{ for } \Omega_{pt,sub} \text{ (cf. (13.6))}$$

$$(23.15) \quad \text{No boundary condition at } (0, 0) \text{ for } \Omega_{sup} \text{ (cf. (13.8))}$$

The operator $\mathcal{L}\varphi$ is understood as in Definition 11.7. Notice that the function $\mathcal{L}\varphi$ is continuous in $\overline{\mathcal{U}}$ and therefore the conditions (23.12)-(23.14) are meaningful.

We summarize the result just obtained as follows:

Proposition 23.6. *The problem (23.10) with Ω_σ as one of the operators*

$$\Omega_{t,sub}, \Omega_{nt,sub}, \Omega_{pt,sub}, \Omega_{sup}$$

is equivalent to solving the PDE problem (23.11) in the class of functions $\varphi \in C(X)$, with $\mathcal{L}\varphi \in C(X)$ with the boundary conditions (23.12), (23.13), (23.14) and (23.15) respectively and where the operator is understood as in Definition 11.7.

We now consider the solvability of the problem (23.11) with boundary conditions (23.12), (23.13), (23.14). This will be made using suitable adaptations of the classical Perron's method (cf. for instance [19]) for harmonic functions for each of the specific problems under consideration. To this end, we will use the solution for the Dirichlet problem in admissible domains obtained in Proposition 20.2.

We can now solve the PDE problems stated in Proposition 23.6.

23.3. Operator $\Omega_{t,sub}$: Solvability of (23.11), (23.12). We first solve (23.11), (23.12). We will prove the following result:

Proposition 23.7. *For any $g \in C(X)$ there exists a unique $\varphi \in C(X)$ which solves (23.11), (23.12).*

Defining a suitable class of subsolutions. In order to prove Proposition 23.7 we need to define a suitable concept of subsolution and supersolutions for (23.11), (23.12) for which the boundary condition at the singular point $(0,0)$ holds. We recall that the space of functions $L_b^\infty(X)$ has been defined in Definition 19.1. It is worth noticing that the definition of sub/supersolutions which will be made in the following involve also the boundary conditions at the singular point, differently from the sub/supersolutions for the operator \mathcal{L} in Definition 19.2 where such boundary conditions at $(0,0)$ were not included. The sub/supersolutions for the operator \mathcal{L} in Definition 19.2 are suitable for comparison arguments in admissible domains, while the sub/supersolutions defined here are suitable to obtain global results in the domain X , in particular well-posedness results.

Definition 23.8. *Suppose that $g \in C(X)$. We will say that a function $\varphi \in L_b^\infty(X)$ such that the limit $\lim_{(x,v) \rightarrow (0,0)} \varphi(x,v) = \varphi(0,0)$ exists, is a subsolution of (23.11), (23.12) if $\varphi(0,0) = g(0,0)$ and for all $\psi \in \mathcal{F}(X)$ (cf. (11.9)) with $\psi \geq 0$ we have:*

$$(23.16) \quad \int (\psi - \lambda \mathcal{L}^*(\psi)) \varphi \leq \int g \psi$$

Given $g \in C(X)$, we will say that $\varphi \in L_b^\infty(X)$ is a supersolution of (23.11), (23.12) if $\varphi(0,0) = g(0,0)$ and for all $\psi \in \mathcal{F}(X)$ with $\psi \geq 0$ we have:

$$(23.17) \quad \int (\psi - \lambda \mathcal{L}^*(\psi)) \varphi \geq \int g \psi$$

Suppose that \mathcal{W} is an open subset of X . We will say that $\varphi \in L_b^\infty(\mathcal{W})$ is a subsolution of (23.11), (23.12) in \mathcal{W} if the following conditions hold: (1) If $(0,0) \in \mathcal{W}$ we have $\varphi(0,0) = g(0,0)$. (2) The inequality (23.16) holds for any function $\psi \in \mathcal{F}(\mathcal{W})$, $\psi \geq 0$.

A key property of sub and supersolutions of (23.11), (23.12) is the following:

Lemma 23.9. *Suppose that $\mathcal{W}_1, \mathcal{W}_2$ are two open subsets of X and that $\varphi_1 \in L_b^\infty(\mathcal{W}_1)$, $\varphi_2 \in L_b^\infty(\mathcal{W}_2)$ are two subsolutions of (23.11), (23.12) in the sense of Definition 23.8 in their respective domains. Then the function φ defined by means of $\varphi = \max\{\varphi_1, \varphi_2\}$ in $\mathcal{W}_1 \cap \mathcal{W}_2$, $\varphi = \varphi_1$ in $\mathcal{W}_1 \setminus (\mathcal{W}_1 \cap \mathcal{W}_2)$, $\varphi = \varphi_2$ in $\mathcal{W}_2 \setminus (\mathcal{W}_1 \cap \mathcal{W}_2)$ is a subsolution of (23.11), (23.12) in $\mathcal{W} = (\mathcal{W}_1 \cup \mathcal{W}_2)$ in the sense of Definition 23.8.*

Suppose that $\mathcal{W}_1, \mathcal{W}_2$ are two open subsets of X and that $\varphi_1 \in L_b^\infty(\mathcal{W}_1)$, $\varphi_2 \in L_b^\infty(\mathcal{W}_2)$ are two supersolutions of (23.11), (23.12) in the sense of Definition 23.8 in their respective domains. Then the function φ defined by means of $\varphi = \min\{\varphi_1, \varphi_2\}$ in $\mathcal{W}_1 \cap \mathcal{W}_2$, $\varphi = \varphi_1$ in $\mathcal{W}_1 \setminus (\mathcal{W}_1 \cap \mathcal{W}_2)$, $\varphi = \varphi_2$ in $\mathcal{W}_2 \setminus (\mathcal{W}_1 \cap \mathcal{W}_2)$ is a supersolution of (23.11), (23.12) in $\mathcal{W} = (\mathcal{W}_1 \cup \mathcal{W}_2)$ in the sense of Definition 23.8.

Remark 23.10. *Notice that the functions φ_1, φ_2 are defined only almost everywhere. Nevertheless it is well known that the function $\max\{\varphi_1, \varphi_2\}$ can be defined as a L_b^∞ function defined also almost everywhere.*

Proof. We will follow different strategies in order to obtain the subsolution inequality for test functions ψ whose support does not intersect $\{x=0\}$ and test functions ψ with support intersecting $\{x=0\}$. Suppose that φ_1, φ_2 are two subsolutions as in the statement of the Lemma. Suppose that $\zeta_\varepsilon(x,v)$ is a C^∞ mollifier with the form:

$$\zeta_\varepsilon(x,v) = \frac{1}{\varepsilon^3} \zeta\left(\frac{x}{\varepsilon}, \frac{v}{\varepsilon^2}\right), \quad \varepsilon > 0, \quad \zeta \geq 0, \quad \int_{\mathbb{R}^2} \zeta = 1, \quad \text{supp}(\zeta) \subset \{0 \leq x \leq 1, |v| \leq 1\}$$

Given any $\delta > 0$ we define:

$$(23.18) \quad \mathcal{Z}_\delta = \{(x, v) \in \mathcal{Z} : \text{dist}((x, v), \partial \mathcal{Z}) \geq \delta\}$$

We fix $\delta > 0$ and we then define functions $\varphi_{1,\varepsilon}, \varphi_{2,\varepsilon}$ in the sets $(\mathcal{W}_1^\pm)_\delta, (\mathcal{W}_2^\pm)_\delta$ (cf. (19.8)) with $2\varepsilon < \delta$ by means of $\varphi_{k,\varepsilon} = \zeta_\varepsilon * \varphi_k, k = 1, 2$. Given $\psi \in \mathcal{F}((\mathcal{W}_k^\pm)_\delta)$ with $\psi \geq 0$ we have:

$$\begin{aligned} & \int_{(\mathcal{W}_k^\pm)_\delta} (\varphi_{k,\varepsilon} - \lambda \mathcal{L}(\varphi_{k,\varepsilon})) \psi \\ &= \int_{(\mathcal{W}_k^\pm)_\delta} (\psi - \lambda \mathcal{L}^*(\psi)) \varphi_{k,\varepsilon} \\ &= \int_{(\mathcal{W}_k^\pm)_\delta} (\psi - \lambda \mathcal{L}^*(\psi)) (\varphi_k * \zeta_\varepsilon) \\ &= \int_{(\mathcal{W}_k^\pm)_\delta} (\psi_\varepsilon - \lambda (\zeta_\varepsilon * \mathcal{L}^*(\psi))) \varphi_k \\ &= \int_{(\mathcal{W}_k^\pm)_\delta} (\psi_\varepsilon - \lambda (\mathcal{L}^*(\psi_\varepsilon))) \varphi_k - \lambda \int_{(\mathcal{W}_k^\pm)_\delta} ((\zeta_\varepsilon * \mathcal{L}^*(\psi)) - (\mathcal{L}^*(\zeta_\varepsilon * \psi))) \varphi_k \end{aligned}$$

It is relevant to remark that the functions ψ_ε are well defined in the sets $(\mathcal{W}_k^\pm)_\delta$ even if these sets have a nonempty intersection with the line $\{x = 0\}$, due to our choice of the mollifiers ζ_ε which are supported in the region $x \geq 0$. On the other hand, the functions ψ_ε do not belong in general to $\mathcal{F}((\mathcal{W}_k)_\delta)$ because the condition (11.5) does not necessarily holds.

Notice that:

$$|(\zeta_\varepsilon * \mathcal{L}^*(\psi)) - (\mathcal{L}^*(\zeta_\varepsilon * \psi))| = |\zeta_\varepsilon * (v \partial_x \psi) - v \partial_x (\zeta_\varepsilon * \psi)|$$

Using then the definition of ζ_ε we obtain:

$$\int_{(\mathcal{W}_k^\pm)_\delta} |\zeta_\varepsilon * (v \partial_x \psi) - v \partial_x (\zeta_\varepsilon * \psi)| \leq C\varepsilon \int_{(\mathcal{W}_k^\pm)_\delta} \psi$$

where we use:

$$\begin{aligned} & \int_{(\mathcal{W}_k^\pm)_\delta} |\zeta_\varepsilon * (v \partial_x \psi) - v \partial_x (\zeta_\varepsilon * \psi)| dx dv \\ &= \int_{(\mathcal{W}_k^\pm)_\delta} \left| \int [\partial_x \zeta_\varepsilon(x - y, v - w) w \psi(y, w) - v \partial_x \zeta_\varepsilon(x - y, v - w) \psi(y, w)] dy dw \right| dx dv \\ &= \int_{(\mathcal{W}_k^\pm)_\delta} \left| \int \partial_x \zeta_\varepsilon(x - y, v - w) (w - v) \psi(y, w) dy dw \right| dx dv \\ &\leq C\varepsilon^2 \int dy dw \psi(y, w) \int_{(\mathcal{W}_k^\pm)_\delta} |\partial_x \zeta_\varepsilon(x - y, v - w)| dx dv \leq \frac{C\varepsilon^2}{\varepsilon} \int dy dw \psi(y, w) \end{aligned}$$

Then:

$$(23.19) \quad \left| \int_{(\mathcal{W}_k^\pm)_\delta} (\varphi_{k,\varepsilon} - \lambda \mathcal{L}(\varphi_{k,\varepsilon})) \psi - \int_{(\mathcal{W}_k^\pm)_\delta} (\psi_\varepsilon - \lambda (\mathcal{L}^*(\psi_\varepsilon))) \varphi_k \right| \leq C\varepsilon \|\varphi_k\|_{L^\infty} \int_{(\mathcal{W}_k^\pm)_\delta} \psi$$

Using that φ_k are subsolutions in \mathcal{W}_k it then follows that:

$$\int_{(\mathcal{W}_k^\pm)_{\frac{\delta}{2}}} (\varphi_{k,\varepsilon} - \lambda \mathcal{L}(\varphi_{k,\varepsilon})) \psi \leq \int_{(\mathcal{W}_k^\pm)_{\frac{\delta}{2}}} g\psi + C\varepsilon \|\varphi_k\|_{L^\infty} \int_{(\mathcal{W}_k^\pm)_{\frac{\delta}{2}}} \psi$$

whence the following pointwise estimate follows:

$$(\varphi_{k,\varepsilon} - \lambda \mathcal{L}(\varphi_{k,\varepsilon})) \leq g + C\varepsilon \|\varphi_k\|_{L^\infty} \quad \text{in } (\mathcal{W}_k)_\delta, \quad k = 1, 2$$

We next obtain the subsolution inequality for test functions ψ supported in $(\mathcal{W}_1^\pm)_\delta \cup (\mathcal{W}_2^\pm)_\delta$ for any $\delta > 0$. Indeed, we can assume that the set $\{\varphi_{1,\varepsilon} = \varphi_{2,\varepsilon}\} \cap [(\mathcal{W}_1^\pm)_\delta \cap (\mathcal{W}_2^\pm)_\delta]$ is non-empty, since otherwise the result would follow trivially. Then, Sard's Lemma (cf. [41]) implies that, for any $\varepsilon > 0$ arbitrarily small, there exists a sequence $\rho_n(\varepsilon) \rightarrow 0$ as $n \rightarrow \infty$, such that the curves $\{\varphi_{1,\varepsilon} = \varphi_{2,\varepsilon} + \rho_n(\varepsilon)\} \subset (\mathcal{W}_1)_\delta \cap (\mathcal{W}_2)_\delta$ are smooth. These curves separate the regions $\{\varphi_{1,\varepsilon} > \varphi_{2,\varepsilon} + \rho_n(\varepsilon)\}$ and $\{\varphi_{1,\varepsilon} < \varphi_{2,\varepsilon} + \rho_n(\varepsilon)\}$. We define functions:

$$\begin{aligned} \varphi_\varepsilon^{(n)} &= \max\{\varphi_{1,\varepsilon}, \varphi_{2,\varepsilon} + \rho_n(\varepsilon)\} \quad \text{in } (\mathcal{W}_1^\pm)_{\frac{\delta}{2}} \cap (\mathcal{W}_2^\pm)_{\frac{\delta}{2}} \\ (23.20) \quad \varphi_\varepsilon^{(n)} &= \varphi_{1,\varepsilon} \quad \text{in } (\mathcal{W}_1^\pm)_{\frac{\delta}{2}} \setminus \left((\mathcal{W}_1^\pm)_{\frac{\delta}{2}} \cap (\mathcal{W}_2^\pm)_{\frac{\delta}{2}} \right) \\ \varphi_\varepsilon^{(n)} &= \varphi_{2,\varepsilon} + \rho_n(\varepsilon) \quad \text{in } (\mathcal{W}_2^\pm)_{\frac{\delta}{2}} \setminus \left((\mathcal{W}_1^\pm)_{\frac{\delta}{2}} \cap (\mathcal{W}_2^\pm)_{\frac{\delta}{2}} \right) \end{aligned}$$

We then compute:

$$\begin{aligned} (23.21) \quad & \int_{(\mathcal{W}_1^\pm)_{\frac{\delta}{2}} \cup (\mathcal{W}_2^\pm)_{\frac{\delta}{2}}} (\psi - \lambda \mathcal{L}^*(\psi)) \varphi_\varepsilon^{(n)} \\ &= \int_{\left[(\mathcal{W}_1^\pm)_{\frac{\delta}{2}} \setminus \left((\mathcal{W}_1^\pm)_{\frac{\delta}{2}} \cap (\mathcal{W}_2^\pm)_{\frac{\delta}{2}} \right) \right] \cup \{\varphi_{1,\varepsilon} > \varphi_{2,\varepsilon} + \rho_n(\varepsilon)\}} (\psi - \lambda \mathcal{L}^*(\psi)) \varphi_{1,\varepsilon} + \\ & \quad + \int_{\left[(\mathcal{W}_2^\pm)_{\frac{\delta}{2}} \setminus \left((\mathcal{W}_1^\pm)_{\frac{\delta}{2}} \cap (\mathcal{W}_2^\pm)_{\frac{\delta}{2}} \right) \right] \cup \{\varphi_{1,\varepsilon} < \varphi_{2,\varepsilon} + \rho_n(\varepsilon)\}} (\psi - \lambda \mathcal{L}^*(\psi)) (\varphi_{2,\varepsilon} + \rho_n(\varepsilon)) \end{aligned}$$

Using the fact that the functions $\varphi_{k,\varepsilon}$ are smooth and integrating by parts we obtain:

$$\begin{aligned} & \int_{\left[(\mathcal{W}_1^\pm)_{\frac{\delta}{2}} \setminus \left((\mathcal{W}_1^\pm)_{\frac{\delta}{2}} \cap (\mathcal{W}_2^\pm)_{\frac{\delta}{2}} \right) \right] \cup \{\varphi_{1,\varepsilon} > \varphi_{2,\varepsilon} + \rho_n(\varepsilon)\}} (\psi - \lambda \mathcal{L}^*(\psi)) \varphi_{1,\varepsilon} \\ &= \int_{\left[(\mathcal{W}_1^\pm)_{\frac{\delta}{2}} \setminus \left((\mathcal{W}_1^\pm)_{\frac{\delta}{2}} \cap (\mathcal{W}_2^\pm)_{\frac{\delta}{2}} \right) \right] \cup \{\varphi_{1,\varepsilon} > \varphi_{2,\varepsilon} + \rho_n(\varepsilon)\}} (\varphi_{1,\varepsilon} - \lambda \mathcal{L}(\varphi_{1,\varepsilon})) \psi + \\ & \quad - \lambda \int_{\partial\{\varphi_{1,\varepsilon} > \varphi_{2,\varepsilon} + \rho_n(\varepsilon)\}} [n_v \varphi_{1,\varepsilon} \partial_v \psi - (n_v \partial_v \varphi_{1,\varepsilon} + v n_x \varphi_{1,\varepsilon}) \psi] ds \end{aligned}$$

where $n = (n_x, n_v)$ is the normal vector to $\partial\{\varphi_{1,\varepsilon} > \varphi_{2,\varepsilon} + \rho_n(\varepsilon)\}$ pointing outwards from the domain $\{\varphi_{1,\varepsilon} > \varphi_{2,\varepsilon} + \rho_n(\varepsilon)\}$. Using a similar argument to compute the last integral in (23.21) we obtain, after some cancellations of terms in the boundary $\partial\{\varphi_{1,\varepsilon} > \varphi_{2,\varepsilon} + \rho_n(\varepsilon)\}$:

$$\begin{aligned} & \int_{(\mathcal{W}_1^\pm)_{\frac{\delta}{2}} \cup (\mathcal{W}_2^\pm)_{\frac{\delta}{2}}} (\psi - \lambda \mathcal{L}^*(\psi)) \varphi_\varepsilon^{(n)} \\ &= \int_{(\mathcal{W}_1^\pm)_{\frac{\delta}{2}} \cup (\mathcal{W}_2^\pm)_{\frac{\delta}{2}}} \left(\varphi_\varepsilon^{(n)} - \lambda \mathcal{L}^*(\varphi_\varepsilon^{(n)}) \right) \psi \\ & \quad + \lambda \int_{\partial\{\varphi_{1,\varepsilon} > \varphi_{2,\varepsilon} + \rho_n(\varepsilon)\}} n_v [\partial_v \varphi_{1,\varepsilon} - \partial_v (\varphi_{2,\varepsilon} + \rho_n(\varepsilon))] \psi ds \end{aligned}$$

where n is the same normal vector as before. Using the pointwise inequality (23.19) as well as the fact that, $n_v \partial_v \varphi_{1,\varepsilon} \leq n_v \partial_v (\varphi_{2\varepsilon} + \rho_n(\varepsilon))$ we obtain:

$$(23.22) \quad \int_{(\mathcal{W}_1^\pm)_{\frac{\delta}{2}} \cup (\mathcal{W}_2^\pm)_{\frac{\delta}{2}}} (\psi - \lambda \mathcal{L}^*(\psi)) \varphi_\varepsilon^{(n)} \leq \int_{(\mathcal{W}_1^\pm)_{\frac{\delta}{2}} \cup (\mathcal{W}_2^\pm)_{\frac{\delta}{2}}} g\psi + C\varepsilon \|\varphi_k\|_{L^\infty} \int_{(\mathcal{W}_1^\pm)_{\frac{\delta}{2}} \cup (\mathcal{W}_2^\pm)_{\frac{\delta}{2}}} \psi$$

for any $\psi \in \mathcal{F}((\mathcal{W}_1^\pm)_\delta \cup (\mathcal{W}_2^\pm)_\delta)$ with $\psi \geq 0$. Notice that these functions might be different from zero in the line $\{x=0\}$ if $((\mathcal{W}_1^\pm)_\delta \cup (\mathcal{W}_2^\pm)_\delta) \cap \{x=0\} \neq \emptyset$. However, they do not satisfy the condition (11.5) because by definition, these functions are only defined in $\{v > -x\}$ or $\{v < x\}$.

We now use Lebesgue's Dominated Convergence Theorem to take the limit $\varepsilon \rightarrow 0$ in (23.22). Therefore, using also the fact that δ can be chosen arbitrarily small, we obtain:

$$(23.23) \quad \int_{\mathcal{W}_1^\pm \cup \mathcal{W}_2^\pm} (\psi - \lambda \mathcal{L}^*(\psi)) \varphi \leq \int_{\mathcal{W}_1^\pm \cup \mathcal{W}_2^\pm} g\psi$$

where $\psi \in \mathcal{F}(\mathcal{W}_1^\pm \cup \mathcal{W}_2^\pm)$ with $\psi \geq 0$ and $\text{supp}(\psi) \cap \{x=0\} = \emptyset$.

It only remains to extend the validity of (23.23) to arbitrary functions $\psi \in \mathcal{F}(\mathcal{W}_1 \cup \mathcal{W}_2)$ with $\psi \geq 0$ (which in particular satisfy (11.5)). To this end we use Proposition 19.3 which proves the existence of traces for subsolutions for the operator \mathcal{L} . Since φ_1, φ_2 are subsolutions (23.11), (23.12) and they are bounded, we obtain that $(-\varphi_1), (-\varphi_2)$ are supersolutions of $\mathcal{L}(\cdot) + \kappa$ for a suitable constant κ in the sense of Definition 19.2 and any admissible domain $\Xi \subset \mathcal{W}_k$ with $k=1,2$ respectively. Therefore we can define φ_1, φ_2 in the sense of traces (see Remark 19.4) at $\{x=0\}$ as $x \rightarrow 0^+$ for $v > 0$ and $v < 0$ respectively. We will denote these limits as $\varphi_{k,+}, \varphi_{k,-}$ with $k=1,2$. Notice that these functions are in the spaces $L^\infty(\{x=0, v > 0\})$ and $L^\infty(\{x=0, v < 0\})$ respectively. Moreover, we now claim that:

$$(23.24) \quad \varphi_{k,+}(0, rv) \geq \varphi_{k,-}(0, -v) \quad , \quad a.e. \ v > 0, \ (0, v) \in \{x=0\} \cap \mathcal{W}_k \quad , \quad k=1,2$$

The inequality (23.24) is equivalent to

$$(23.25) \quad \int_{\{v>0\}} \varphi_{k,+}(0, r\cdot) \zeta(\cdot) \geq \int_{\{v>0\}} \varphi_{k,-}(0, -\cdot) \zeta(\cdot)$$

for any $\zeta = \zeta(v) \geq 0$, $\zeta \in C^\infty(\{v > 0\})$. To prove (23.25) we argue as follows. Proposition 19.3 implies that $\lim_{x \rightarrow 0^+} \int_{\{v>0\}} \varphi_{k,\pm}(x, r\cdot) \zeta(\cdot) = \int_{\{v>0\}} \varphi_{k,\pm}(0, r\cdot) \zeta(\cdot)$. We construct a function $\psi \in \mathcal{F}(\mathcal{W}_k)$ as follows. We define a function $\eta = \eta(\frac{x}{\varepsilon})$ with $\varepsilon > 0$, $\eta \in C^\infty(\{x \geq 0\})$, $\eta' \leq 0$, $\eta(x) = 0$ if $x \geq 1$, $\eta(0) = 1$. We then define $\psi(x, v) = \frac{\zeta(\frac{v}{r})\eta(x)}{rv}$ for $v > 0$ and $\psi(x, v) = \frac{\zeta(-v)\eta(x)}{|v|}$ for $v < 0$. Notice that $r^2\psi(0^+, rv) = \psi(0^+, -v) = \frac{\zeta(v)\eta(x)}{v}$ for $v > 0$. Therefore $\psi \in \mathcal{F}(\mathcal{W}_k)$ (cf. (11.9)). Integrating by parts in (23.16)

$$\int_{\mathcal{W}_k} (\psi - \lambda D_v^2 \psi - \lambda v D_x \psi) \varphi_k + \lambda \int_{\{x=0, v>0\}} v \varphi_{k,+} \frac{\zeta(\frac{\cdot}{r})}{rv} + \lambda \int_{\{x=0, v<0\}} v \varphi_{k,-} \frac{\zeta(v)}{|v|} \leq \int g\psi$$

Taking the limit $\varepsilon \rightarrow 0$ we obtain $\frac{1}{r} \int_{\{x=0, v>0\}} \varphi_{k,+} \zeta(\frac{\cdot}{r}) - \int_{\{x=0, v<0\}} \varphi_{k,-} \zeta(v) \geq 0$, whence, after using the change of variables $\frac{\cdot}{r} \rightarrow \cdot$ in the first integral we obtain (23.25) whence (23.24) follows.

We now claim that it is possible to define the limit values $\varphi_+ = \varphi(0^+, v)$, $\varphi_- = \varphi(0^+, -v)$ for $v > 0$ in the sense of traces, with $\varphi = \max\{\varphi_1, \varphi_2\}$. Indeed, this is a consequence of the fact that φ satisfies the subsolution inequality (23.16) for any test function $\psi \geq 0$ with $\text{supp}(\psi) \cap \{x=0\} \neq \emptyset$. Therefore, the argument yielding Proposition 19.3 implies the existence of φ_+, φ_- .

We now use Lemmas 19.6, 19.10 as well as the inequalities (23.24) to prove that:

$$\begin{aligned}\varphi^-(0, -v) &= \max \{ \varphi_1^-(0, -v), \varphi_2^-(0, -v) \} \\ &\leq \max \{ \varphi_1^+(0, rv), \varphi_2^+(0, rv) \} \leq (\max \{ \varphi_1(0, rv), \varphi_2(0, rv) \})^+ = \varphi^+(0, rv)\end{aligned}$$

for *a.e.* $v > 0$. We can then argue as in the derivation of (23.24) to show that φ satisfies also the subsolution inequality at the line $\{x = 0\}$. This shows that φ is a subsolution and concludes the proof of the result. \square

The main idea in Perron's method is that it is possible to construct a solution of the problem if we can obtain one subsolution φ^{sub} and a supersolution φ^{sup} satisfying $\varphi^{sub} \leq \varphi^{sup}$. Such sub and supersolutions can be easily obtained in the case of the problem (23.12), (23.13) and Definition 23.8:

Lemma 23.11. *For any $g \in C(X)$ there exist at least one subsolution φ^{sub} and one supersolution φ^{sup} in the sense of Definition 23.8 such that:*

$$(23.26) \quad \varphi^{sub} \leq \varphi^{sup}$$

Proof. Let $|g| \leq \|g\|_{L^\infty(X)}$ and let $\varphi^{sup} = \min \{ g(0, 0) + KF_\beta + \|g\|_{L^\infty(X)} S, \|g\|_{L^\infty(X)} \}$, $\varphi^{sub} = \max \{ g(0, 0) - KF_\beta - \|g\|_{L^\infty(X)} S, -\|g\|_{L^\infty(X)} \}$, where $S(x, v)$ is a super-solution with $\mathcal{L}S = 1$, constructed in Lemma 21.1 and $K > 0$ must be determined. We have that $g(0, 0) + KF_\beta(x, v) + \|g\|_{L^\infty(X)} S(x, v) > 0$ in a neighbourhood of the singular point which will be denoted as \mathcal{W}_1 , if we take $K > 0$ sufficiently large. Moreover, this function is a supersolution of (23.11), (23.12) in \mathcal{W}_1 by construction. On the other hand $\|g\|_{L^\infty(X)}$ is a supersolution of (23.11), (23.12) in an open set \mathcal{W}_2 such that $\mathcal{W}_1 \cup \mathcal{W}_2 = X$ and $\mathcal{W}_1 \cap \mathcal{W}_2 \neq \emptyset$. Therefore φ^{sup} is a supersolution due to Lemma 23.9. To prove that φ^{sub} is a subsolution we use a similar argument. \square

We define the following subset of $L_b^\infty(X)$:

Definition 23.12. *Suppose that φ^{sub} , φ^{sup} are respectively one subsolution and one supersolution in the sense of Definition 23.8 satisfying (23.26). We then define $\tilde{\mathcal{G}}_{sub} \subset L_b^\infty(X)$ as:*

$$(23.27) \quad \tilde{\mathcal{G}}_{sub} \equiv \left\{ \varphi \in L_b^\infty(X) : \begin{array}{l} \varphi \text{ sub-solution of (23.11), (23.12)} \\ \text{in the sense of Definition 23.8} \mid \varphi^{sub} \leq \varphi \leq \varphi^{sup} \end{array} \right\}.$$

Notice that we have:

Lemma 23.13. *The set $\tilde{\mathcal{G}}_{sub}$ is closed in the weak topology, defined by means of the functionals $\ell_\psi(\varphi) = \int_X \varphi \psi$ with $\psi \in \mathcal{F}(X)$ (cf. (11.9)).*

Proof. It is just a consequence of the definition 23.8, as well as the fact that the inequalities $\varphi^{sub} \leq \varphi \leq \varphi^{sup}$ are preserved by weak limits. \square

Remark 23.14. *We recall that the bounded set $\tilde{\mathcal{G}}_{sub}$ endowed with the weak-* topology is metrizable (cf. [10]). We will denote the corresponding metric as dist_* .*

The following Lemma will be useful in order to show that the supremum of the set $\tilde{\mathcal{G}}_{sub}$ can be obtained by means of limits of subsequences.

Lemma 23.15. *There exists a countable and dense subset \mathcal{G}_{sub} of $\tilde{\mathcal{G}}_{sub}$ in the weak topology in Lemma 23.13.*

Proof. We will prove the result by showing that there exists a countable subset of $\tilde{\mathcal{G}}_{sub}$ which is dense in the $L^2_{loc}(X)$ topology. Since the test functions in $\mathcal{F}(X)$ are compactly supported, this is enough to prove the required density property in the weak topology. We find a countable subset \mathcal{G}_{sub} of $\tilde{\mathcal{G}}_{sub}$ by using the fact that $L^2(K)$ is separable for any compact subsets K : Choose a sequence of compact sets $\{K_n\}_{n=1}^\infty$ such that $X = \cup_{n=1}^\infty K_n$, $K_n \subset K_{n+1}$. For each K_n , $L^2(K_n) = \cup_{k_N=1}^\infty B_{1/N}(f_{k_N})$, where $f_{k_N} \in L^2(K_n)$, $k_N = 1, 2, \dots$ and the distance is measured in L^2 norm. Then we choose one element from each $B_{1/N}(f_{k_N})$ if $\tilde{\mathcal{G}}_{sub} \cap B_{1/N}(f_{k_N})$ is nonempty and choose none if $\tilde{\mathcal{G}}_{sub} \cap B_{1/N}(f_{k_N})$ is empty. This selection can be made for each $N = 1, 2, 3, \dots$ and then for $n = 1, 2, 3, \dots$. We call this subset \mathcal{G}_{sub} . This set is countable and dense in $\tilde{\mathcal{G}}_{sub}$ in L^2 norm. This completes the proof. \square

Next we show that the set $\tilde{\mathcal{G}}_{sub}$ is closed under the maximum function.

Lemma 23.16. *Let $\varphi_1, \varphi_2, \dots, \varphi_L \in \tilde{\mathcal{G}}_{sub}$ with $L < \infty$ and $\tilde{\mathcal{G}}_{sub}$ as in Definition 24.8. Define:*

$$\bar{\varphi} := \max\{\varphi_1, \varphi_2, \dots, \varphi_L\}$$

Then $\bar{\varphi} \in \tilde{\mathcal{G}}_{sub}$.

Proof. It is just a consequence of Lemma 23.9. \square

We want to give a definition of the largest subsolution in the set $\tilde{\mathcal{G}}_{sub}$. Given that the functions φ in $\tilde{\mathcal{G}}_{sub}$ are not defined pointwise we cannot just take the supremum $\sup_{\tilde{\mathcal{G}}_{sub}} \varphi$. However, the function that will play the role of such supremum is the following. We can assume that the countable family \mathcal{G}_{sub} constructed in Lemma 23.15 is $\{\varphi_j\}_{j \in \mathbb{N}}$. We then construct the following finite families of subsolutions:

$$\mathcal{G}_{sub}(M) = \{\varphi_j\}_{j=1}^M$$

and we then define the following function which will play the role of $\sup_{\tilde{\mathcal{G}}_{sub}} \varphi$:

$$(23.28) \quad \varphi_* = \lim_{M \rightarrow \infty} \max_{\mathcal{G}_{sub}(M)} \varphi$$

Notice that $\max_{\mathcal{G}_{sub}(M)} \varphi$ is the maximum of a finite number of functions and therefore is a well defined quantity. On the other hand, the sequence $\{\max_{\mathcal{G}_{sub}(M)} \varphi\}$ is increasing in M and uniformly bounded by φ^{\sup} , therefore the limit on the right-hand side of (23.28) exists in L^1_{loc} and then also in the weak topology defined in Lemma 23.13.

Our next goal is to show that φ_* is the desired solution of the problem (23.11), (23.12). To this end we need an auxiliary result, namely the solvability of the Dirichlet problem in the admissible domains defined in Definition 12.1 with boundary values in the admissible boundaries.

End of the Proof of Proposition 23.7. The following result will be used to prove that if a subsolution is not a solution of (23.11), (23.12), it is possible to construct a larger subsolutions.

Lemma 23.17. *Suppose that $\bar{\varphi}$ is a subsolution of (23.11), (23.12) in the sense of Definition 23.8. Let Ξ be any admissible domain in the sense of Definition 12.1 contained in X . Let us denote as h the boundary values of $\bar{\varphi}$ in $\partial_a \Xi$ obtained from the interior of Ξ (cf. Proposition 19.3). Let φ be the unique solution of (20.1), (20.2) obtained in Proposition 20.2. We construct a function Φ by means of:*

$$(23.29) \quad \Phi = \begin{cases} \bar{\varphi}, & (x, v) \notin X \setminus \Xi \\ \varphi, & (x, v) \in \Xi \end{cases}$$

Then Φ is a sub-solution of (23.11), (23.12) in the sense of Definition 23.8.

Proof. It follows using some arguments analogous to those in the Proof of Lemma 23.9, and therefore we will skip the details of the proof. Notice that the function Φ is larger than $\bar{\varphi}$ in Ξ . Therefore the contribution of the terms arising at the boundary of Ξ in the definition of subsolution, including at the possible points of discontinuity of Φ , have the sign required for Φ to be a subsolution. \square

End of the Proof of Proposition 23.7. We define φ_* as in (23.28). The limit in (23.28) is pointwise a.e in X . Due to Lebesgue's Theorem it then follows that φ_M converges to φ_* in the weak topology introduced in Lemma 23.13. Applying this Lemma, as well as the fact that the definition of $\tilde{\mathcal{G}}_{sub}$ (cf. (23.27)) and φ^{sub} , φ^{sup} (cf. Lemma 23.11) it follows that φ_* is a subsolution of (23.11), (23.12) with $\varphi_* \in \tilde{\mathcal{G}}_{sub}$. This implies that $\nu = -(\varphi_* - \lambda\mathcal{L}(\varphi_*) - g)$ defines a nonnegative Radon measure. If $\nu = 0$, it follows that φ_* is a solution of (23.11), (23.12) and the Proposition would easily follow. Suppose then that $\nu \neq 0$. Then, there exists an admissible domain $\Xi \subset X$ such that $\nu(S) > 0$ with $S = \left\{ (x, v) : \text{dist}((x, v), \partial\Xi) \geq \frac{d(\Xi)}{4} \right\}$ with $d(\Xi)$ as in (20.7). Suppose that φ is the unique solution of (20.1), (20.2) obtained in Proposition 20.2 where h is the trace of φ_* in $\partial_a\Xi$ obtained from the interior of Ξ . We then define Φ as in (23.29) with $\bar{\varphi} = \varphi_*$. Lemma 23.17 implies that Φ is a subsolution of (23.11), (23.12) in the sense of Definition 23.8. On the other hand, since $\varphi_* \in \tilde{\mathcal{G}}_{sub}$ we have $h \leq \varphi^{sup}$ on $\partial_a\Xi$, where φ^{sup} is understood in $\partial_a\Xi$ in the sense of trace from the interior of Ξ . Then $\varphi \leq \varphi^{sup}$ due to Proposition 20.6, whence $\Phi \leq \varphi^{sup}$. Therefore $\Phi \in \tilde{\mathcal{G}}_{sub}$. Due to Lemma 20.4 and using that $\nu(S) > 0$ we have that $\Phi > \varphi_* + \delta$ for some $\delta > 0$ in an open subset of Ξ . Since $\varphi_* \geq \varphi_j$ for any $\varphi_j \in \mathcal{G}_{sub}$ with \mathcal{G}_{sub} as in Lemma 23.15 it then follows that $\text{dist}_*(\Phi, \mathcal{G}_{sub}) > 0$, where dist_* is the metric associated to the weak-* topology used in Lemma 23.13 (cf. Remark 23.14). However, Lemma 23.15 implies that \mathcal{G}_{sub} is dense in $\tilde{\mathcal{G}}_{sub}$ in the weak-* topology, and therefore this gives a contradiction. Then $\nu = 0$ and thus $\varphi_* - \lambda\mathcal{L}(\varphi_*) - g = 0$ in the sense of distributions. Since φ_* is bounded, we can use Theorem 12.4 to prove that $\varphi_* \in W^{1,p}(U)$ in any bounded set $U \subset X$ whose closure does not intersect $(0, 0)$ and any $1 < p < \infty$. Therefore φ_* is continuous away from the origin. Since $\varphi_* \in \tilde{\mathcal{G}}_{sub}$ it also follows that φ_* is continuous at $(0, 0)$. It remains to check that φ_* is also continuous at the point ∞ . Notice that our choice of sub/supersolutions imply that $|\varphi_*| \leq \|g\|_{L^\infty(X)}$. We can now prove that $\varphi_*(x, v) \rightarrow g(\infty)$ as $(x, v) \rightarrow \infty$ as follows. In any large domain contained in the regions $\{v > 0\}$ or $\{v < 0\}$ we then obtain, using parabolic theory, that φ_* converges to $g(\infty)$. Indeed, in the case of $\{v < 0\}$ we just integrate the parabolic equation in the direction of increasing x and it readily follows that $\varphi_*(x, v)$ approaches to $g(\infty)$. If $\{v > 0\}$, we argue similarly, but with decreasing x . It then follows also, due to the boundedness of $|\varphi_*|$ that φ_* converges to $g(\infty)$ if $|(x, v)| \rightarrow \infty$ and $v \rightarrow \infty$. Therefore $\varphi_*(x, v) \rightarrow g(\infty)$ if $|(x, v)| \rightarrow \infty$ and $|v| \rightarrow \infty$. In order to prove the convergence for $x \rightarrow \infty$ and $|v|$ bounded we argue as follows. After subtracting from φ_* quantities like $g(\infty) \pm \varepsilon$, with $\varepsilon > 0$ it follows that we only need to show that for large admissible domains Ξ with bounded values of ψ at the admissible boundary $\partial_a\Xi$ and ψ satisfying $\psi - \lambda\mathcal{L}(\psi) = 0$ we have $|\psi(x, v)|$ small if the size of Ξ tends to infinity and the distance from (x, v) to $\partial_a\Xi$ tends to infinity too. This follows from instance from (20.5) using the fact that $\tau(\Xi)$ tends to infinity for the points under consideration with a large probability, and estimating the remainder by the small probability of having $\tau(\Xi)$ of order one.

Therefore $\varphi_*(\infty) = g(\infty)$ and the result follows. \square

24. OPERATOR $\Omega_{nt,sub}$. SOLVABILITY OF (23.11), (23.13).

We now argue as in the previous case in order to show that the problem (23.11), (23.13) can be solved for any $g \in C(X)$. The main difference arises in the analysis of the behaviour

of the solution φ in a neighbourhood of the singular point $(0,0)$ and in particular showing that (23.13) holds. The key point will be to show that for the maximum of a suitable family of subsolutions $\tilde{\mathcal{G}}_{sub}$, we have that $\mathcal{A}(\varphi)$ is well defined and $\mathcal{A}(\varphi) \geq 0$. If $\mathcal{A}(\varphi) > 0$, it turns out that it is possible to construct a larger subsolution in the family $\tilde{\mathcal{G}}_{sub}$. This contradiction will imply $\mathcal{A}(\varphi) = 0$.

Several arguments needed to prove the solvability of the problem (23.11), (23.13) are similar to the ones in the previous Section. We will emphasize just the points where differences arise. The main result that we prove in this Section is the following.

Proposition 24.1. *For any $g \in C(X)$ there exists a unique $\varphi \in C(X)$ which solves (23.11), (23.13).*

We will prove Proposition 24.1 using a method analogous to the proof of Proposition 23.7. However we need to define a different class of subsolutions, which allows us to identify the boundary condition (23.13). The main novelty is that the test functions might have a power law singularity near the singular point $(x, v) = (0, 0)$.

Definition 24.2. *Suppose that $g \in C(X)$. Let $\zeta = \zeta(x, v)$ be a nonnegative C^∞ test function supported in $0 \leq x + |v|^3 \leq 2$, satisfying $\zeta = 1$ for $0 \leq x + |v|^3 \leq 1$. We will say that a function $\varphi \in L_b^\infty(X)$ is a subsolution of (23.11), (23.13) if for all ψ with $\psi \geq 0$, $\psi(0, -v) = r^2\psi(0, rv)$, $v > 0$ and having the form $\psi = \theta\zeta G_\gamma + \bar{\psi}$ where $\bar{\psi} \in C^\infty$, $\theta \geq 0$ with G_γ as in (4.8), $\gamma \in \{-\frac{2}{3}, \alpha\}$ and with ψ supported in a set contained in the ball $|(x, v)| \leq R$ for some $R > 0$ we have:*

$$(24.1) \quad \int (\psi - \lambda \mathcal{L}^*(\psi)) \varphi \leq \int g \psi$$

Given $g \in C(X)$, we will say that $\varphi \in L_b^\infty(X)$ is a supersolution of (23.11), (23.13) if for all ψ with the same properties as above we have:

$$(24.2) \quad \int (\psi - \lambda \mathcal{L}^*(\psi)) \varphi \geq \int g \psi$$

Notice that there are two differences between the Definitions 23.8 and 24.2. In this second definition we do not impose $\varphi(0, 0) = g(0, 0)$. On the other hand, we have a larger class of admissible test functions in Definition 24.2. Notice that the integral on the left hand side of (24.1) is well defined in spite of the singularity of G_γ near the singular point because $\mathcal{L}^*(G_\gamma) = 0$.

Our next goal is to prove the analogous of Lemma 23.9 for the operator $\Omega_{nt, sub}$. This will require to prove that it is possible to define in a suitable sense the quantity $\mathcal{A}(\varphi)$ (cf. (13.1)) for sub or supersolutions in the sense of Definition 24.2. This will be made by means of the limit as $\delta \rightarrow 0^+$ of the quantity $\Psi(\delta)$ defined in the next Lemma.

Lemma 24.3. *Let $g \in C(X)$ and φ be a subsolution of (23.11), (23.13) in the sense of Definition 24.2. Let:*

$$(24.3) \quad \Psi(\delta) = \int_{\partial \mathcal{R}_\delta} (n_v G_\alpha \partial_v \varphi - n_v \partial_v G_\alpha \varphi + v \varphi G_\alpha n_x) ds$$

which is well defined in the sense of Traces (cf. Remark 19.4). Here $n = (n_x, n_v)$ is the unit normal vector pointing toward \mathcal{R}_δ . Then the limit $\lim_{\delta \rightarrow 0^+} \Psi(\delta)$ exists and we have:

$$\lim_{\delta \rightarrow 0^+} \Psi(\delta) \leq 0$$

Let $\mu = -\varphi + \lambda \mathcal{L}(\varphi) + g$. Then $\mu \geq 0$ defines a Radon measure in $X \setminus \{(0, 0)\}$ satisfying:

$$(24.4) \quad \int_{\mathcal{R}_1 \setminus \{(0, 0)\}} G_\alpha \mu dx dv < \infty$$

Proof. Using test functions $\psi \geq 0$, with support disjoint from $(x, v) = (0, 0)$ we obtain that (24.1) implies the existence of a measure $\mu \geq 0$ such that:

$$(24.5) \quad -\varphi + \lambda \mathcal{L}(\varphi) + g = \mu \geq 0$$

Notice that part of the support of μ can be contained in the line $\{x = 0\}$. Notice that it might be possible to have $\mu(\mathbb{R}^+ \times \mathbb{R}) = \infty$.

Suppose that φ is a subsolution of (23.11), (23.13) in the sense of Definition 24.2. Multiplying (24.5) by G_α and integrating in domains $\mathcal{R}_{\delta_2} \setminus \mathcal{R}_{\delta_1}$ with $\delta_1 < \delta_2$ as well as the fact that the traces of the function φ and some of the derivatives $\partial_v \varphi$ exist, we obtain:

$$(24.6) \quad \begin{aligned} & -\Psi(\delta_2) + \Psi(\delta_1) \\ &= \frac{1}{\lambda} \int_{\mathcal{R}_{\delta_2} \setminus \mathcal{R}_{\delta_1}} G_\alpha \mu dx dv + \frac{1}{\lambda} \int_{\mathcal{R}_{\delta_2} \setminus \mathcal{R}_{\delta_1}} G_\alpha \varphi dx dv - \frac{1}{\lambda} \int_{\mathcal{R}_{\delta_2} \setminus \mathcal{R}_{\delta_1}} G_\alpha g dx dv, \end{aligned}$$

where $\Psi(\delta)$ is as in (24.3). Notice that the existence of the traces of φ , $\partial_v \varphi$ in the required boundaries $\partial \mathcal{R}_{\delta_1}$, $\partial \mathcal{R}_{\delta_2}$ due to Proposition 19.3. Notice that the proof of (24.6) must be made approximating the characteristic function of the domain $\mathcal{R}_{\delta_2} \setminus \mathcal{R}_{\delta_1}$ by smooth functions and taking the limit. Combining (24.6) with the fact that $\mu \geq 0$, and φ and g are bounded implies:

$$(24.7) \quad -\Psi(\delta_2) + \Psi(\delta_1) \geq -\frac{C}{\lambda} \int_{\mathcal{R}_{\delta_2} \setminus \mathcal{R}_{\delta_1}} G_\alpha dx dv$$

We can compute the integrals $\int_{\mathcal{R}_\delta} G_\alpha dx dv$. Indeed, using Proposition 4.2 and (4.27) we obtain:

$$\int_{\mathcal{R}_\delta} G_\alpha dx dv = \int_{\mathcal{R}_\delta} x^\alpha \Phi\left(-\frac{v^3}{9x}\right) dx dv = A \delta^{3(1+\alpha)+1}$$

where $A = \int_{-1}^r dv \int_0^1 x^\alpha \Phi\left(-\frac{v^3}{9x}\right) dx$. It is readily seen that $0 < A < \infty$ and also that $3(1+\alpha) > 0$. The inequality (24.7) then implies that the function $\Psi(\delta) - \frac{CA\delta^{3(1+\alpha)+1}}{\lambda}$ is decreasing. Then the limit $\lim_{\delta \rightarrow 0^+} \left(\Psi(\delta) - \frac{A\delta^{3(1+\alpha)+1}}{\lambda}\right)$ exists, whence the limit $\lim_{\delta \rightarrow 0^+} \Psi(\delta)$ exists too, where the value of this limit might be ∞ . Our goal is to show that:

$$(24.8) \quad \lim_{\delta \rightarrow 0^+} \Psi(\delta) \leq 0$$

We will consider separately the cases $\lim_{\delta \rightarrow 0^+} \Psi(\delta) < \infty$ and $\lim_{\delta \rightarrow 0^+} \Psi(\delta) = \infty$. Suppose first that $\lim_{\delta \rightarrow 0^+} \Psi(\delta) < \infty$. Using (24.6) with $\delta_2 = 1$, and taking the limit $\delta_1 \rightarrow 0$ we obtain:

$$\begin{aligned} & -\Psi(1) + \Psi(0^+) \\ &= \frac{1}{\lambda} \int_{\mathcal{R}_1 \setminus \{(0,0)\}} G_\alpha \mu dx dv + \frac{1}{\lambda} \int_{\mathcal{R}_1 \setminus \{(0,0)\}} G_\alpha \varphi dx dv - \frac{1}{\lambda} \int_{\mathcal{R}_1 \setminus \{(0,0)\}} G_\alpha g dx dv. \end{aligned}$$

The boundedness of φ implies that $\int_{\mathcal{R}_1 \setminus \{(0,0)\}} G_\alpha \varphi dx dv = \int_{\mathcal{R}_1} G_\alpha \varphi dx dv$ is also bounded. Then, using the fact that $\mu \geq 0$ and $G_\alpha > 0$ we obtain:

$$(24.9) \quad 0 \leq \int_{\mathcal{R}_1 \setminus \{(0,0)\}} G_\alpha \mu dx dv < \infty$$

We now argue as follows. We choose a test function $\psi = G_\alpha \xi$, with $\xi = 1$ in $\overline{\mathcal{R}_\delta}$ and $\xi = 0$ in $X \setminus \mathcal{R}_{2\delta}$, with $\delta > 0$, $\xi \geq 0$. Using Definition 24.2 we obtain (24.5). Notice that $\mathcal{L}^*(\psi)$ is integrable near the origin and φ is bounded. Then:

$$(24.10) \quad \int ((\psi - \lambda \mathcal{L}^*(\psi)) \varphi - g\psi) = \lim_{\delta \rightarrow 0} \int_{X \setminus \mathcal{R}_\delta} ((\psi - \lambda \mathcal{L}^*(\psi)) \varphi - g\psi)$$

Integrating by parts and using (24.5), (24.3) we obtain:

$$\begin{aligned} & \int_{X \setminus \mathcal{R}_\delta} ((\psi - \lambda \mathcal{L}^*(\psi)) \varphi - g\psi) \\ &= \lambda \int_{\partial \mathcal{R}_\delta} (n_v G_\alpha \partial_v \varphi - n_v \partial_v G_\alpha \varphi + v \varphi G_\alpha n_x) ds + \\ & \quad - \int_{X \setminus \mathcal{R}_\delta} \psi \mu dx dv = \lambda \Psi(\delta) - \int_{X \setminus \mathcal{R}_\delta} \psi \mu dx dv \end{aligned}$$

Using Definition 24.2 and (24.10) we arrive at:

$$\lim_{\delta \rightarrow 0} \left[\lambda \Psi(\delta) - \int_{X \setminus \mathcal{R}_\delta} \psi \mu dx dv \right] \leq 0$$

and (24.9) yields $\lim_{\delta \rightarrow 0} \int_{X \setminus \mathcal{R}_\delta} \psi \mu dx dv = 0$, whence (24.8) follows if $\lim_{\delta \rightarrow 0^+} \Psi(\delta) < \infty$.

Suppose now that $\lim_{\delta \rightarrow 0^+} \Psi(\delta) = \infty$. Then $\Psi(\delta) > 0$ can be made arbitrarily large for δ small. We consider test functions with the form $\psi = G_\alpha \xi$ where $\xi \geq 0$ is a function which takes constant values in each set $\partial \mathcal{R}_\rho$ for each $\rho > 0$. Moreover, we will assume that $\xi = \bar{\xi}(\rho)$ satisfies $\bar{\xi}'(\rho) \leq 0$ for ρ small, and $\bar{\xi}'$ globally bounded. We will assume that $\bar{\xi}$ is constant for small ρ , whence ψ is an admissible test function. We have $\int (\psi - \lambda \mathcal{L}^*(\psi)) \varphi = \lim_{\delta \rightarrow 0} \int_{X \setminus \mathcal{R}_\delta} (\psi - \lambda \mathcal{L}^*(\psi)) \varphi$. Integrating by parts we obtain:

$$\int_{X \setminus \mathcal{R}_\delta} (\psi - \lambda \mathcal{L}^*(\psi)) \varphi = -\lambda \int_{\partial \mathcal{R}_\delta} n_v D_v \psi \varphi ds + \int_{X \setminus \mathcal{R}_\delta} (\psi \varphi + \lambda (D_v \psi D_v \varphi + v D_x \psi \varphi))$$

Using then that $\psi = G_\alpha \xi$ we rewrite this expression as:

$$\begin{aligned} & \int_{X \setminus \mathcal{R}_\delta} (\psi - \lambda \mathcal{L}^*(\psi)) \varphi \\ &= -\lambda \int_{\partial \mathcal{R}_\delta} n_v (\xi D_v G_\alpha + G_\alpha D_v \xi) \varphi ds + \\ & \quad + \int_{X \setminus \mathcal{R}_\delta} (\psi \varphi + \lambda (\xi D_v G_\alpha D_v \varphi + G_\alpha D_v \xi D_v \varphi + v D_x G_\alpha \xi \varphi + v G_\alpha D_x \xi \varphi)) \end{aligned}$$

We now use that $D_{vv} G_\alpha = v D_x G_\alpha$ to rewrite this formula as:

$$\begin{aligned} & \int_{X \setminus \mathcal{R}_\delta} (\psi - \lambda \mathcal{L}^*(\psi)) \varphi \\ &= -\lambda \int_{\partial \mathcal{R}_\delta} n_v (\xi D_v G_\alpha + G_\alpha D_v \xi) \varphi ds + \\ & \quad + \int_{X \setminus \mathcal{R}_\delta} (\psi \varphi + \lambda (\xi D_v G_\alpha D_v \varphi + G_\alpha D_v \xi D_v \varphi + D_{vv} G_\alpha \xi \varphi + v G_\alpha D_x \xi \varphi)) \end{aligned}$$

Then, integrating by parts in the term containing $D_{vv} G_\alpha$ we obtain:

$$\begin{aligned} & \int_{X \setminus \mathcal{R}_\delta} (\psi - \lambda \mathcal{L}^*(\psi)) \varphi \\ &= -\lambda \int_{\partial \mathcal{R}_\delta} n_v (\xi D_v G_\alpha + G_\alpha D_v \xi) \varphi ds + \lambda \int_{\partial \mathcal{R}_\delta} n_v \xi D_v G_\alpha \varphi ds + \\ & \quad + \int_{X \setminus \mathcal{R}_\delta} (\psi \varphi + \lambda (\xi D_v G_\alpha D_v \varphi + G_\alpha D_v \xi D_v \varphi - D_v G_\alpha D_v (\xi \varphi) + v G_\alpha D_x \xi \varphi)) \end{aligned}$$

whence, after some cancellations and rearrangements of terms:

$$(24.11) \quad \int_{X \setminus \mathcal{R}_\delta} (\psi - \lambda \mathcal{L}^*(\psi)) \varphi = -\lambda \int_{\partial \mathcal{R}_\delta} n_v G_\alpha D_v \xi \varphi ds + \int_{X \setminus \mathcal{R}_\delta} (\psi \varphi + \lambda Q)$$

where:

$$Q = D_v \xi [G_\alpha D_v \varphi - D_v G_\alpha \varphi] + v G_\alpha D_x \xi \varphi$$

We now use the following Fubini's like formula:

$$(24.12) \quad \int_{X \setminus \mathcal{R}_\delta} (\psi \varphi + \lambda Q) = \int_{X \setminus \mathcal{R}_\delta} \psi \varphi + \lambda \int_\delta^\infty d\rho \int_{\partial \mathcal{R}_\rho} J_\rho Q ds_\rho$$

where ds_ρ is the arc-length in $\partial \mathcal{R}_\rho$ and J_ρ is obtained by means of the formula:

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_{\mathcal{R}_{\rho+h} \setminus \mathcal{R}_\rho} \omega dx dv = \int_{\partial \mathcal{R}_\rho} J_\rho \omega ds_\rho$$

for any continuous test function ω . It is readily seen that the function J_ρ is well defined by means of elementary arguments. Moreover, it is given by:

$$J_\rho = \left\{ \begin{array}{l} 1, \quad x \in [0, \rho^3], \quad v = -\rho \\ r, \quad x \in [0, \rho^3], \quad v = r\rho \\ 3\rho^2, \quad x = \rho^3, \quad -\rho \leq v \leq r\rho \end{array} \right\}$$

On the other hand, using our choice of ξ we obtain:

$$D_x \xi = \frac{\bar{\xi}'(\rho)}{3\rho^2} \text{ if } -x \leq v^3 \leq r^3 x, \quad D_x \xi = 0 \text{ if } v^3 < -x \text{ or } v^3 > r^3 x$$

$$D_v \xi = -\bar{\xi}'(\rho) \text{ if } v^3 < -x, \quad D_v \xi = r\bar{\xi}'(\rho) \text{ if } v^3 > r^3 x, \quad D_v \xi = 0 \text{ if } -x \leq v^3 \leq r^3 x$$

Then, using also that the normal vector (n_x, n_v) points towards \mathcal{R}_ρ we obtain:

$$J_\rho Q = -\bar{\xi}'(\rho) [n_v [G_\alpha D_v \varphi - D_v G_\alpha \varphi] + v \varphi G_\alpha n_x]$$

Using this formula in (24.11), (24.12) we obtain:

$$\begin{aligned} \int_{X \setminus \mathcal{R}_\delta} (\psi - \lambda \mathcal{L}^*(\psi)) \varphi &= -\lambda \int_{\partial \mathcal{R}_\delta} n_v G_\alpha D_v \xi \varphi ds + \int_{X \setminus \mathcal{R}_\delta} \psi \varphi - \\ &\quad -\lambda \int_\delta^\infty \bar{\xi}'(\rho) d\rho \int_{\partial \mathcal{R}_\rho} [n_v [G_\alpha D_v \varphi - D_v G_\alpha \varphi] + v \varphi G_\alpha n_x] ds_\rho \end{aligned}$$

whence, using (24.3) as well as the fact that $\bar{\xi}'(\rho) = 0$ if ρ is sufficiently small, we obtain:

$$\int_{X \setminus \mathcal{R}_\delta} (\psi - \lambda \mathcal{L}^*(\psi)) \varphi = \int_{X \setminus \mathcal{R}_\delta} \psi \varphi - \lambda \int_\delta^\infty \bar{\xi}'(\rho) \Psi(\rho) d\rho$$

if δ is sufficiently small. Notice that $\Psi(\rho)$ is unbounded as $\rho \rightarrow 0$. Choosing $\bar{\xi}'(\rho) < 0$ for small ρ , but globally bounded we can make $-\int_\delta^\infty \bar{\xi}'(\rho) \Psi(\rho) d\rho$ sufficiently large. On the other hand, we have $\lim_{\delta \rightarrow 0} \int_{X \setminus \mathcal{R}_\delta} (\psi - \lambda \mathcal{L}^*(\psi)) \varphi = \int (\psi - \lambda \mathcal{L}^*(\psi)) \varphi \leq \int g\psi \leq C$ and $-\int_{X \setminus \mathcal{R}_\delta} \psi \varphi \leq C$ for some suitable constant C independent of δ . Then:

$$-\int_\delta^\infty \bar{\xi}'(\rho) \Psi(\rho) d\rho \leq \frac{3C}{\lambda}$$

for any $\bar{\xi} \geq 0$ bounded. However, as indicated above, it is possible to choose $\bar{\xi}$, δ yielding $-\int_\delta^\infty \bar{\xi}'(\rho) \Psi(\rho) d\rho$ arbitrarily large with $\bar{\xi} \geq 0$ bounded. This contradiction then yields $\lim_{\delta \rightarrow 0^+} \Psi(\delta) < \infty$ whence the Lemma follows. \square

We will show now also that it is possible to define the limit value $\varphi(0, 0)$ for sub/supersolutions in the sense of Definition 24.2. This will be made by means of the limit as $\delta \rightarrow 0$ of the quantity $H(\delta)$ defined below.

Lemma 24.4. *Assume $r < r_c$. Let $g \in C(X)$ and φ a subsolution of (23.11), (23.13) in the sense of Definition 24.2. Suppose that the normal vector (n_x, n_v) points towards \mathcal{R}_δ . Let:*

$$H(\delta) = \int_{\partial\mathcal{R}_\delta} \left(n_v G_{-\frac{2}{3}} \partial_v \varphi - n_v \partial_v G_{-\frac{2}{3}} \varphi + v \varphi G_{-\frac{2}{3}} n_x \right) ds$$

which is well defined in the sense of Traces (cf. Proposition 19.3). Then the limit $\lim_{\delta \rightarrow 0^+} H(\delta) = H(0)$ exists. We also have:

$$(24.13) \quad \lim_{\delta \rightarrow 0^+} \varphi(\delta^3 x, \delta v) = -\frac{H(0)}{9^{\frac{2}{3}} \left[\log(r) + \frac{\pi}{\sqrt{3}} \right]} = \varphi(0, 0)$$

in L_{loc}^p for some $p > 1$.

Proof. We use in Definition 24.2 the test function $\psi = \xi G_{-\frac{2}{3}}$ where $\xi = 1$ for $(x, v) \in \mathcal{R}_{\delta_2} \setminus \mathcal{R}_{\delta_1}$ and $\xi = 0$ otherwise. We then obtain:

$$\begin{aligned} -H(\delta_2) + H(\delta_1) &= \frac{1}{\lambda} \int_{\mathcal{R}_{\delta_2} \setminus \mathcal{R}_{\delta_1}} G_{-\frac{2}{3}} \mu dx dv + \\ &\quad + \frac{1}{\lambda} \int_{\mathcal{R}_{\delta_2} \setminus \mathcal{R}_{\delta_1}} G_{-\frac{2}{3}} \varphi dx dv - \frac{1}{\lambda} \int_{\mathcal{R}_{\delta_2} \setminus \mathcal{R}_{\delta_1}} G_{-\frac{2}{3}} g dx dv \end{aligned}$$

Using (24.4) combined with the fact that $\alpha < -\frac{2}{3}$ we obtain $0 \leq \frac{1}{\lambda} \int_{\mathcal{R}_{\delta_2} \setminus \mathcal{R}_{\delta_1}} G_{-\frac{2}{3}} \mu dx dv \leq C(\delta_2)^b$ for $b = -2 - 3\alpha > 0$. Then, using the boundedness of φ and g we obtain $|H(\delta_2) - H(\delta_1)| \leq C(\delta_2)^b$. Choosing $\delta_1 = \frac{\delta_2}{2}$ and iterating, we obtain that the limit $\lim_{\delta \rightarrow 0} H(\delta)$ exists and it is bounded.

In order to prove the convergence of $\varphi_\delta(x, v) = \varphi(\delta^3 x, \delta v)$ we notice that the functions φ_δ satisfy:

$$(24.14) \quad -\delta^2 \varphi_\delta + \lambda \mathcal{L}(\varphi_\delta) + \delta^2 g_\delta = \mu_\delta$$

where:

$$\mu_\delta(x, v) = \delta^2 \mu(\delta^3 x, \delta v) \quad \text{and} \quad g_\delta(x, v) = \delta^2 g(\delta^3 x, \delta v).$$

Notice that (24.4) yields:

$$\int_{\mathcal{R}_1 \setminus \{(0,0)\}} G_{-\frac{2}{3}} \mu_\delta dx dv \leq C(\delta)^b$$

The measures μ_δ converge to zero in the weak topology in compact sets. On the other hand we have:

$$H(\delta) = \int_{\partial\mathcal{R}_1} \left(n_v G_{-\frac{2}{3}} \partial_v \varphi_\delta - n_v \partial_v G_{-\frac{2}{3}} \varphi_\delta + v \varphi_\delta G_{-\frac{2}{3}} n_x \right) ds$$

The functions φ_δ are bounded. Taking suitable subsequences we obtain convergence to a limit $\hat{\varphi}$ in the weak topology. Taking the limit in (24.14) we obtain that $\hat{\varphi}$ solves the following equation in the sense of distributions:

$$\mathcal{L}(\hat{\varphi}) = 0$$

The traces of the functions φ_δ at the boundaries of $\partial\mathcal{R}_1$ are defined uniformly in δ , in the sense that the derivatives of the integrals of φ_δ in vertical lines (and horizontal lines with some derivatives), have uniformly bounded derivatives (in L^1 norms), with derivatives converging to zero as $\delta \rightarrow 0$ (cf. Lemma 19.11). Then, we can take the limit of $H(\delta)$ to obtain:

$$(24.15) \quad H(0^+) = \int_{\partial\mathcal{R}_1} \left(n_v G_{-\frac{2}{3}} \partial_v \hat{\varphi} - n_v \partial_v G_{-\frac{2}{3}} \hat{\varphi} + v \hat{\varphi} G_{-\frac{2}{3}} n_x \right) ds$$

Moreover, the same argument implies that $\hat{\varphi}$ satisfies (11.1). The point (iv) in Theorem 21.2 then implies that $\hat{\varphi}$ is constant (since it is bounded). We will denote this constant as $\varphi(0, 0)$. Due to (24.15) the limit is independent of the subsequence and we obtain, using (4.28):

$$H(0^+) = -9^{\frac{2}{3}}\varphi(0, 0) \left[\log(r) + \frac{\pi}{\sqrt{3}} \right]$$

The convergence takes place in L^p for some $p > 1$, due to the regularizing effects for hypoelliptic operators in Theorem 12.4. \square

The following computation shows that the subsolutions of (23.11), (23.13) in the sense of Definition 24.2 with the asymptotics (13.1) must have a restriction on the sign of $\mathcal{A}(\varphi)$.

Lemma 24.5. *Suppose that φ is a subsolution of (23.11), (23.13) in the sense of Definition 24.2. Let us assume also that φ satisfies (13.1). Then:*

$$(24.16) \quad \lim_{\delta \rightarrow 0^+} \Psi(\delta) = C_* \mathcal{A}(\varphi)$$

where $\Psi(\delta)$ is as in (24.3) and C_* is as in Proposition 10.3. Moreover, we have $\mathcal{A}(\varphi) \geq 0$.

Proof. Taking the limit $\delta \rightarrow 0$ and using (13.1), we can approximate the right-hand side of (24.3) as:

$$\mathcal{A}(\varphi) \int_{\partial \mathcal{R}_\delta} [n_v G_\alpha D_v F_\beta - n_v D_v G_\alpha F_\beta + v F_\beta G_\alpha n_x] ds \rightarrow C_* \mathcal{A}(\varphi) \text{ as } \delta \rightarrow 0,$$

where we have used (10.11). This gives (24.16). We recall that $C_* < 0$, and since φ is a subsolution of (23.11), (23.13) in the sense of Definition 24.2, by Lemma 24.3, we obtain that $\mathcal{A}(\varphi) \geq 0$ whence the result follows. \square

Lemma 24.6. *Suppose that $\mathcal{W}_1, \mathcal{W}_2$ are two open subsets of X and that $\varphi_1 \in L_b^\infty(\mathcal{W}_1)$, $\varphi_2 \in L_b^\infty(\mathcal{W}_2)$ are two subsolutions of (23.11), (23.13) in the sense of Definition 24.2 in their respective domains. Then the function φ defined by means of $\varphi = \max\{\varphi_1, \varphi_2\}$ in $\mathcal{W}_1 \cap \mathcal{W}_2$, $\varphi = \varphi_1$ in $\mathcal{W}_1 \setminus (\mathcal{W}_1 \cap \mathcal{W}_2)$, $\varphi = \varphi_2$ in $\mathcal{W}_2 \setminus (\mathcal{W}_1 \cap \mathcal{W}_2)$ is a subsolution of (23.11), (23.13) in $\mathcal{W} = (\mathcal{W}_1 \cup \mathcal{W}_2)$ in the sense of Definition 24.2.*

Suppose that $\mathcal{W}_1, \mathcal{W}_2$ are two open subsets of X and that $\varphi_1 \in L_b^\infty(\mathcal{W}_1)$, $\varphi_2 \in L_b^\infty(\mathcal{W}_2)$ are two supersolutions of (23.11), (23.13) in the sense of Definition 24.2 in their respective domains. Then the function φ defined by means of $\varphi = \min\{\varphi_1, \varphi_2\}$ in $\mathcal{W}_1 \cap \mathcal{W}_2$, $\varphi = \varphi_1$ in $\mathcal{W}_1 \setminus (\mathcal{W}_1 \cap \mathcal{W}_2)$, $\varphi = \varphi_2$ in $\mathcal{W}_2 \setminus (\mathcal{W}_1 \cap \mathcal{W}_2)$ is a supersolution of (23.11), (23.13) in $\mathcal{W} = (\mathcal{W}_1 \cup \mathcal{W}_2)$ in the sense of Definition 24.2.

Proof. It is enough to prove the result for subsolutions, since for supersolutions the argument is similar. We recall that we use test functions with the form $\psi = \theta \zeta G_\alpha + \bar{\psi}$ where $\bar{\psi} \in C^\infty$, $\theta \geq 0$ with G_α as in (4.8). If $\theta = 0$ and the support of $\bar{\psi}$ does not intersect the singular point the inequality (24.1) follows arguing exactly as in the Proof of Lemma 23.9. On the other hand, since φ is bounded, we can use a limit argument as well as the fact that the contributions of the integrals $\int \mathcal{L}^*(\bar{\psi}) \varphi$ in the region close to the origin are small if $\bar{\psi}$ is smooth near the singular point to prove (24.1) for $\bar{\psi} \in C^\infty$ compactly supported. The only difficulty is to derive the inequality for $\psi = \theta \zeta G_\alpha + \bar{\psi}$ with $\theta > 0$ and ζ as in Definition 24.2. We will assume in the following that $\theta = 1$.

We define $\varphi = \max\{\varphi_1, \varphi_2\}$. We need to prove that $\int (\psi - \lambda \mathcal{L}^*(\psi)) \varphi - \int g\psi \leq 0$ for ψ as above. Notice that:

$$\int (\psi - \lambda \mathcal{L}^*(\psi)) \varphi - \int g\psi = \lim_{\varepsilon \rightarrow 0^+} \left[\int_{X \setminus \mathcal{R}_\varepsilon} (\psi - \lambda \mathcal{L}^*(\psi)) \varphi - \int_{X \setminus \mathcal{R}_\varepsilon} g\psi \right]$$

Integrating by parts and using Proposition 19.3 we obtain for $\varepsilon > 0$:

$$\begin{aligned} & \int_{X \setminus \mathcal{R}_\varepsilon} (\psi - \lambda \mathcal{L}^*(\psi)) \varphi - \int_{X \setminus \mathcal{R}_\varepsilon} g \psi \\ &= \lambda \int_{\partial \mathcal{R}_\varepsilon} (n_v G_\alpha \partial_v \varphi - n_v \partial_v G_\alpha \varphi + v \varphi G_\alpha n_x) ds + \\ & \quad + \int_{X \setminus \mathcal{R}_\varepsilon} \psi (\varphi - \lambda \mathcal{L}(\varphi)) - \int_{X \setminus \mathcal{R}_\varepsilon} g \psi \end{aligned}$$

We have that $(\varphi - \lambda \mathcal{L}(\varphi)) - g \leq 0$ in the sense of measures as it might be seen arguing as in the proof of Lemma 23.9 in sets which do not contain the singular point $(0, 0)$. We then obtain:

$$(24.17) \quad \int_{X \setminus \mathcal{R}_\varepsilon} (\psi - \lambda \mathcal{L}^*(\psi)) \varphi - \int_{X \setminus \mathcal{R}_\varepsilon} g \psi \leq \lambda \int_{\partial \mathcal{R}_\varepsilon} (n_v G_\alpha \partial_v \varphi - n_v \partial_v G_\alpha \varphi + v \varphi G_\alpha n_x) ds$$

We now need to examine the sign of the right-hand side of (24.17). To this end we first prove the existence of $\varphi(0, 0)$ in some suitable sense. Using that $-\mu = (\varphi - \lambda \mathcal{L}(\varphi)) - g \leq 0$ we obtain, after multiplying by $G_{-\frac{2}{3}}$ and integrating by parts in the domain $\mathcal{R}_{\varepsilon_2} \setminus \mathcal{R}_{\varepsilon_1}$ with $\varepsilon_1 < \varepsilon_2$ sufficiently small:

$$\Phi(\varepsilon_2) - \Phi(\varepsilon_1) \leq \int_{\mathcal{R}_{\varepsilon_2} \setminus \mathcal{R}_{\varepsilon_1}} (g - \varphi) G_{-\frac{2}{3}}, \quad \varepsilon_1 < \varepsilon_2$$

where:

$$\Phi(\varepsilon) = \lambda \int_{\partial \mathcal{R}_\varepsilon} \left(n_v G_{-\frac{2}{3}} \partial_v \varphi - n_v \partial_v G_{-\frac{2}{3}} \varphi + v \varphi G_{-\frac{2}{3}} n_x \right) ds$$

and where n is as usual the normal vector pointing towards the origin. It then follows that $\Phi(\varepsilon) + C\varepsilon$ is decreasing for a suitable constant C and ε sufficiently small. We have now two possibilities, either $\lim_{\varepsilon \rightarrow 0} \Phi(\varepsilon) = \infty$ or $\lim_{\varepsilon \rightarrow 0} \Phi(\varepsilon) < \infty$. We now claim that the first possibility contradicts the boundedness of φ . Indeed, to check this we argue as follows. Suppose that $\lim_{\varepsilon \rightarrow 0} \Phi(\varepsilon) = \infty$. We define $\tilde{\varphi} = \varphi + R - m$ where $m = \sup_{\mathcal{R}_1} (\varphi + R)$ and where R satisfies $-\lambda \mathcal{L}(R) \leq \|g - \varphi\|_\infty$ in order to have $-\lambda \mathcal{L}(\tilde{\varphi}) \leq -\mu$. Notice that R can be chosen as a quadratic function in v near the singular point as in Lemma 21.1. Then $\tilde{\varphi} \leq 0$ in \mathcal{R}_1 . We define a sequence $\mu_k = \int_{\mathcal{R}_{2^{-k}} \setminus \mathcal{R}_{2^{-(k+1)}}} \mu G_{-\frac{2}{3}}$. We remark that $\lim_{\varepsilon \rightarrow 0} \Phi(\varepsilon) = \infty$ is equivalent to $\sum_k^\infty \mu_k = \infty$. Notice that:

$$-\lambda \mathcal{L}(\tilde{\varphi}) \leq -\mu$$

We can obtain an upper estimate for $\tilde{\varphi}$ as follows. We define functions $\tilde{\varphi}_\ell$ as the solutions of:

$$\begin{aligned} \lambda \mathcal{L}(\tilde{\varphi}_\ell) &= \mu \chi_{\mathcal{R}_{2^{-\ell}} \setminus \mathcal{R}_{2^{-(\ell+1)}}} \quad \text{in } \mathcal{R}_1 \setminus \mathcal{R}_{2^{-m}}, \ell + 1 < m, \ell > 1 \\ \tilde{\varphi}_\ell &= 0 \quad \text{in } \partial_a(\mathcal{R}_1 \setminus \mathcal{R}_{2^{-m}}) \end{aligned}$$

Since $\tilde{\varphi}_\ell \leq 0$ we can compare with the function $\hat{\varphi}_\ell$ which solves:

$$\begin{aligned} \lambda \mathcal{L}(\hat{\varphi}_\ell) &= \mu \chi_{\mathcal{R}_{2^{-\ell}} \setminus \mathcal{R}_{2^{-(\ell+1)}}} \quad \text{in } \mathcal{R}_{2^{-(\ell-1)}} \setminus \mathcal{R}_{2^{-(\ell+2)}}, \ell + 1 < m, \ell > 1 \\ \hat{\varphi}_\ell &= 0 \quad \text{in } \partial_a(\mathcal{R}_{2^{-(\ell-1)}} \setminus \mathcal{R}_{2^{-(\ell+2)}}) \end{aligned}$$

Then $\tilde{\varphi}_\ell \leq \hat{\varphi}_\ell$ in $\mathcal{R}_{2^{-(\ell-1)}} \setminus \mathcal{R}_{2^{-(\ell+2)}}$. Proposition 20.6 combined with a rescaling argument yields $\hat{\varphi}_\ell \leq -c_0 \mu_\ell$ in $\partial \mathcal{R}_{2^{-\ell}}$ for some $c_0 > 0$ independent on ℓ . Then $\tilde{\varphi}_\ell \leq -c_0 \mu_\ell$ in $\partial \mathcal{R}_{2^{-\ell}}$. On the other hand we have $\tilde{\varphi}_\ell = 0$ in $\partial_a \mathcal{R}_1$. We can then use comparison with a function with the form $w_\ell = -c_1 \mu_\ell (1 - KF_\beta)$ with K chosen large enough and c_1 sufficiently small to guarantee that $w_\ell \geq 0$ in $\partial_a \mathcal{R}_1$ and $w_\ell \geq \tilde{\varphi}_\ell$ in $\partial_a \mathcal{R}_{2^{-\ell}}$. This implies $w_\ell \geq \tilde{\varphi}_\ell$ in $\mathcal{R}_1 \setminus \mathcal{R}_{2^{-\ell}}$. Therefore, since K is independent of ℓ , it follows that $\tilde{\varphi}_\ell \leq -\frac{c_1}{2} \mu_\ell$ in $\mathcal{R}_{2^{-J}} \setminus \mathcal{R}_{2^{-\ell}}$ for some $J > 0$ independent of ℓ . We now use that $\tilde{\varphi} \leq \sum_{\ell=J}^m \tilde{\varphi}_\ell$ for any m . Then $\tilde{\varphi} \leq -\frac{c_1}{2} \sum_{\ell=J}^m \mu_\ell$

in $\mathcal{R}_{2-J} \setminus \mathcal{R}_{2-(J+1)}$ and since $\sum_k^\infty \mu_k = \infty$ it then follows that $\tilde{\varphi} = -\infty$ in $\mathcal{R}_{2-J} \setminus \mathcal{R}_{2-(J+1)}$. However, we know that $\tilde{\varphi}$ is bounded. This contradiction yields $\sum_k^\infty \mu_k < \infty$. Then $\mu_k \rightarrow 0$ as $k \rightarrow \infty$. We can then argue as in the derivation of (24.13) to prove that the limit $\lim_{\delta \rightarrow 0^+} \varphi(\delta^3 x, \delta v) = \varphi(0, 0)$ exists in L_{loc}^p for some $p > 1$.

We now claim that $\varphi(0, 0) = \max\{\varphi_1(0, 0), \varphi_2(0, 0)\}$. Suppose without loss of generality that $\varphi_2(0, 0) \geq \varphi_1(0, 0)$. We then have $\varphi(\delta^3 x, \delta v) \geq \varphi_2(\delta^3 x, \delta v)$ whence $\varphi(0, 0) \geq \varphi_2(0, 0)$. On the other hand we have $\varphi_1(\delta^3 x, \delta v) = \varphi_1(0, 0) + \varepsilon_{1,\delta}(x, v)$, $\varphi_2(\delta^3 x, \delta v) = \varphi_2(0, 0) + \varepsilon_{2,\delta}(x, v)$, where $\varepsilon_{k,\delta}(x, v) \rightarrow 0$ as $\delta \rightarrow 0$ in L_{loc}^p , $k = 1, 2$. Then $\varphi(\delta^3 x, \delta v) \leq \varphi_2(0, 0) + |\varepsilon_{1,\delta}(x, v)| + |\varepsilon_{2,\delta}(x, v)|$, whence $\varphi(0, 0) \leq \varphi_2(0, 0)$ follows. Therefore $\varphi(0, 0) = \max\{\varphi_1(0, 0), \varphi_2(0, 0)\}$. Arguing now as in the Proof of Lemma 24.3 we obtain that there exists the limit $\lim_{\delta \rightarrow 0^+} \Psi(\delta)$ with $\Psi(\delta)$ defined as in (24.3). This limit might take perhaps the value $+\infty$. We will denote this limit as $\Psi_\varphi(0)$ in order to make explicit the dependence on the function φ . Lemma 24.3 implies the existence of the corresponding limits $\Psi_{\varphi_1}(0)$, $\Psi_{\varphi_2}(0)$ for the subsolutions φ_1 , φ_2 respectively. Moreover, we have $\Psi_{\varphi_1}(0) \leq 0$, $\Psi_{\varphi_2}(0) \leq 0$.

We now distinguish two different cases. Suppose first that $\varphi_1(0, 0) < \varphi_2(0, 0)$, where the values $\varphi_k(0, 0)$, $k = 1, 2$ are defined as in (24.13). We then argue as follows. Taking into account (24.4) we obtain that $\varepsilon_{k,\delta}(x, v) = |\varphi_k(\delta^3 x, \delta v) - \varphi_k(0, 0)|$, $k = 1, 2$ satisfies:

$$\|\varepsilon_k\|_{L^p(K)} < C(\delta)^{-3\alpha-2} \quad \text{as } \delta \rightarrow 0^+$$

for some $p > 1$. This estimate follows from (24.4) using a rescaling argument in the equation satisfied by φ_k .

Notice that $-3\alpha - 2 > 0$. We have also that:

$$-\varepsilon_1 \leq \varphi(\delta^3 x, \delta v) - \varphi_2(\delta^3 x, \delta v) \leq \varepsilon_2$$

whence:

$$(24.18) \quad \|\varphi(\delta^3 \cdot, \delta \cdot) - \varphi_2(\delta^3 \cdot, \delta \cdot)\|_{L^p(K)} < C(\delta)^{-3\alpha-2}$$

We now study the behaviour of the function $\Psi_\varphi(\delta)$ as $\delta \rightarrow 0$. We denote as φ^δ the rescaled function $\varphi^\delta = \varphi(\delta^3 \cdot, \delta \cdot)$ and we define similarly φ_k^δ for $k = 1, 2$. Then it follows that:

$$(24.19) \quad \Psi_\varphi(\delta\zeta) = \delta^{2+3\alpha} \int_{\partial\mathcal{R}_\zeta} \left(n_v G_\alpha \partial_v \varphi^\delta - n_v \partial_v G_\alpha \varphi^\delta + v \varphi^\delta G_\alpha n_x \right) ds$$

with $\zeta \in [\frac{1}{2}, 2]$. We now need to approximate this integral by the one in which φ^δ is replaced by φ_2^δ . This requires to obtain some continuity of the integrand of the right-hand side of (24.19) using (24.18). This requires to obtain estimates for $(\partial_v \varphi^\delta - \partial_v \varphi_2^\delta)$. Since we need these estimates only in the part of the boundaries $\partial\mathcal{R}_\zeta$ where $|n_v| = 1$ and $n_x = 0$ we can obtain the result using classical parabolic estimates. Notice that the problem reduces to obtaining estimates for the derivatives with respect to v of the solutions of $\mathcal{L}(w) = \nu$ where ν is a measure. Standard theory yields estimates of the form $\int |\partial_v w| dx dv \leq C \|\nu\|$, in bounded sets. It then follows, combining this estimate with (24.18), (24.19) as well as Fubini's Theorem that $\int_{[\frac{1}{2}, 2]} |\Psi_\varphi(\delta\zeta) - \Psi_{\varphi_2}(\delta\zeta)| d\zeta \rightarrow 0$ as $\delta \rightarrow 0$. Therefore, there exists $\zeta = \zeta_\delta$ such that $|\Psi_\varphi(\delta\zeta_\delta) - \Psi_{\varphi_2}(\delta\zeta_\delta)| \rightarrow 0$ as $\delta \rightarrow 0$, whence $\Psi_\varphi(0) = \Psi_{\varphi_2}(0)$. In this case, we can use (24.17) to obtain, taking the limit $\varepsilon \rightarrow 0$ that:

$$\int_{X \setminus \{0,0\}} (\psi - \lambda \mathcal{L}^*(\psi)) \varphi - \int_{X \setminus \{0,0\}} g\psi \leq \lambda \Psi_{\varphi_2}(0)$$

Using then Lemma 24.3 we obtain $\Psi_{\varphi_2}(0) \leq 0$, whence the Subsolution inequality (24.1) follows.

The case $\varphi_1(0,0) > \varphi_2(0,0)$ is similar. Suppose then that $\varphi_1(0,0) = \varphi_2(0,0)$. Lemma 24.3 implies the existence of the limits $\Psi_{\varphi_1}(0)$, $\Psi_{\varphi_2}(0)$. Notice that we have also the existence of $\Psi_\varphi(0)$ although this limit can take the value ∞ .

We define rescaled functions as $\Phi_k^\delta(x, v) = \frac{\varphi_k(\delta^3 x, \delta v) - \varphi_k(0,0)}{\delta^{3\beta}}$, $k = 1, 2$. We then have, using Lemma 24.5, that Φ_k^δ converges to $\mathcal{A}(\varphi_k)F_\beta$ with $\mathcal{A}(\varphi_k) = \frac{1}{C_*} \Psi_{\varphi_k}(0)$. The convergence takes place in L_{loc}^p with $p > 1$. Moreover, arguing as in the previous case we obtain also convergence of $\partial_v \Phi_k^\delta$ to $\mathcal{A}(\varphi_k) \partial_v F_\beta$ in L_{loc}^p . We now have two possibilities. Either $\Psi_{\varphi_1}(0) \neq \Psi_{\varphi_2}(0)$ or $\Psi_{\varphi_1}(0) = \Psi_{\varphi_2}(0)$. In the first case, suppose without loss of generality that $\Psi_{\varphi_1}(0) > \Psi_{\varphi_2}(0)$. We recall that $\varphi = \max\{\varphi_1, \varphi_2\}$. We define also $\Phi^\delta(x, v) = \frac{\varphi(\delta^3 x, \delta v) - \varphi(0,0)}{\delta^{3\beta}}$. Notice that $\varphi_1(0,0) = \varphi_2(0,0) = \varphi_k(0,0)$. We then have, arguing as in the case $\varphi_1(0,0) \neq \varphi_2(0,0)$ that Φ^δ converges to Φ_1^δ in L_{loc}^p . Using the continuity of the functionals $\Psi_\varphi(0)$ on the functions φ obtained above, it then follows that $\Psi_\varphi(0) = \Psi_{\varphi_1}(0) \leq 0$ whence the inequality (24.1) follows also in this case. It remains to examine the case $\Psi_{\varphi_1}(0) = \Psi_{\varphi_2}(0)$. In this case we have that the functions Φ_k^δ for $k = 1, 2$ converge to $\mathcal{A}(\varphi_1)F_\beta = \mathcal{A}(\varphi_2)F_\beta$ in L_{loc}^p . Then $\Phi^\delta = \max\{\Phi_1^\delta, \Phi_2^\delta\}$ converges to the same limit in L_{loc}^p . We have also convergence for the derivatives $\partial_v \Phi^\delta$ using the regularizing effects as usual. Then, using the continuity of the functionals $\Psi_\varphi(0)$ with respect to this topology we obtain $\Psi_\varphi(0) = \Psi_{\varphi_1}(0) = \Psi_{\varphi_2}(0)$, whence $\Psi_\varphi(0) \leq 0$ and (24.1) follows. Therefore φ is a subsolution and the result follows. \square

We can now conclude the proof of Proposition 24.1, arguing as in the case of trapping boundary conditions.

We can obtain in this case easily one subsolution and one supersolution which are ordered.

Lemma 24.7. *For any $g \in C(X)$ there exist at least one subsolution φ^{sub} and one supersolution φ^{sup} in the sense of Definition 24.2 such that:*

$$(24.20) \quad \varphi^{sub} \leq \varphi^{sup}$$

Proof. We can just take $\varphi^{sub} = -\|g\|_{L^\infty(X)}$, $\varphi^{sup} = \|g\|_{L^\infty(X)}$. We need to prove that $\int \mathcal{L}^*(\psi) \varphi \leq 0$ if φ is a constant and ψ is any test function as in the Definition 24.2. We can assume that $\varphi = 1$. We write $\int \mathcal{L}^*(\psi) = \lim_{\delta \rightarrow 0^+} \int_{X \setminus \mathcal{R}_\delta} \mathcal{L}^*(\psi)$ with \mathcal{R}_δ as in (4.27). Integrating by parts and using that ψ is compactly supported, as well as the fact that $\psi(0, -v) = r^2 \psi(0, rv)$, $v > 0$ we obtain:

$$\int_{X \setminus \mathcal{R}_\delta} \mathcal{L}^*(\psi) = \int_{\partial \mathcal{R}_\delta} [D_v \psi n_v - v \psi n_x]$$

where $n = (n_x, n_v)$ is the normal vector to $\partial \mathcal{R}_\delta$ pointing towards \mathcal{R}_δ and where $\psi = \theta G_\gamma + \bar{\psi}$ if δ is sufficiently small and $\gamma \in \{-\frac{2}{3}, \alpha\}$. The regularity of $\bar{\psi}$ implies that

$$\lim_{\delta \rightarrow 0^+} \int_{\partial \mathcal{R}_\delta} [D_v \bar{\psi} n_v - v \bar{\psi} n_x] = 0.$$

On the other hand we have

$$\lim_{\delta \rightarrow 0^+} \int_{\partial \mathcal{R}_\delta} [D_v G_\alpha n_v - v G_\alpha n_x] = 0$$

due to Proposition 4.11. Finally we notice that

$$\lim_{\delta \rightarrow 0^+} \int_{\partial \mathcal{R}_\delta} [D_v G_{-\frac{2}{3}} n_v - v G_{-\frac{2}{3}} n_x] \leq 0$$

due to Proposition 4.8 whence the result follows. \square

We can then define $\tilde{\mathcal{G}}_{sub}$ as in Definition the following subset of $L_b^\infty(X)$:

Definition 24.8. Suppose that φ^{sub} , φ^{sup} are respectively one subsolution and one supersolution in the sense of Definition 24.2 satisfying (24.20). We then define $\tilde{\mathcal{G}}_{sub} \subset L_b^\infty(X)$ as:

$$(24.21) \quad \tilde{\mathcal{G}}_{sub} \equiv \left\{ \varphi \in L_b^\infty(X) : \begin{array}{l} \varphi \text{ sub-solution of (23.11), (23.13)} \\ \text{in the sense of Definition 24.2} \mid \varphi^{sub} \leq \varphi \leq \varphi^{sup} \end{array} \right\}.$$

We now remark that Lemmas 23.13 and 23.15 hold without any changes. Moreover, the proof of Lemma 23.16 can be adapted to include this case.

Lemma 24.9. Let $\varphi_1, \varphi_2, \dots, \varphi_L \in \tilde{\mathcal{G}}_{sub}$ with $L < \infty$ and $\tilde{\mathcal{G}}_{sub}$ as in Definition 24.8. Define:

$$\bar{\varphi} := \max\{\varphi_1, \varphi_2, \dots, \varphi_L\}$$

Then $\bar{\varphi} \in \tilde{\mathcal{G}}_{sub}$.

Proof. It is enough to prove the result for $L = 2$. The result then follows from Lemma 24.6. Notice that the inequalities $\varphi^{sub} \leq \varphi \leq \varphi^{sup}$ in Definition (24.21) are preserved by the maxima. \square

We remark that Proposition 20.2 and Lemma 20.4 are independent of the boundary conditions imposed at the singular point $(x, v) = (0, 0)$ and therefore can be applied to subsolutions as in Definition 24.8. Lemma 23.17 can be proved for the subsolutions in the class $\tilde{\mathcal{G}}_{sub}$ given in Definition 24.8 with minor cases with respect to the previous case, because the definition of Φ in (23.29) modifies the subsolution $\bar{\varphi}$ just in a set Ξ which does not intersect the singular point $(0, 0)$. We can now conclude the Proof of Proposition 24.1.

End of the Proof of Proposition 24.1. We define φ_* as in (23.28) with $\tilde{\mathcal{G}}_{sub}$ as in Definition 24.8. Arguing as in the Proof of Proposition 23.7 we obtain that $\varphi_* \in \tilde{\mathcal{G}}_{sub}$ with $\tilde{\mathcal{G}}_{sub}$ as in Definition 24.8. We define $\nu = -(\varphi_* - \lambda\mathcal{L}(\varphi_*) - g)$. Due to Definition 24.2 we have that ν is a nonnegative Radon measure. If $\nu(X \setminus \{0, 0\}) > 0$ we can argue exactly as in the Proof of Proposition 23.7 to derive a contradiction. Therefore $\nu(X \setminus \{0, 0\}) = 0$ and then, φ_* satisfies the problem:

$$\varphi_* - \lambda\mathcal{L}(\varphi_*) - g = 0, \quad (x, v) \in X \setminus \{0, 0\}, \quad \varphi_* \in C(X)$$

Theorem 21.11 then implies:

$$\varphi_*(x, v) = \varphi_*(0, 0) + \mathcal{A}(\varphi_*)F_\beta(x, v) + \psi_*(x, v)$$

Using Lemma 24.3 and Lemma 24.5 we obtain that $\mathcal{A}(\varphi_*) \geq 0$. Suppose that $\mathcal{A}(\varphi_*) > 0$. Then $\varphi_*(x, v) \geq \varphi_*(0, 0)$ in a neighbourhood of the singular point $(0, 0)$. Then $\varphi_*(0, 0) < \varphi^{sup}$, where φ^{sup} is as in Definition 24.8, because otherwise we would have $\varphi_* = \varphi^{sup}$ and then $\mathcal{A}(\varphi_*) = 0$. Therefore, there exists $\delta > 0$ such that $\varphi_{**} = \max\{\varphi_*, \varphi_*(0, 0) + \delta + CS(x, v)\}$ with $S(x, v)$ as in Lemma 21.1 and $C > 0$ is a subsolution of (23.11), (23.13) in a neighbourhood of the singular point satisfying $\varphi_{**} > \varphi_*$, since S is bounded by $\left(x^{\frac{2}{3}} + v^2\right)$ and therefore $\mathcal{A}(\varphi)F_\beta$ gives a larger contribution for (x, v) close to $(0, 0)$. However, this would contradict the definition of φ_* by means of (23.28). It then follows from this contradiction that $\mathcal{A}(\varphi_*) = 0$. Therefore φ_* solves (23.11), (23.13) and the Proposition follows. \square

25. OPERATOR $\Omega_{pt, sub}$. SOLVABILITY OF (23.11), (23.14).

We consider now the problem (23.11), (23.14). Since the arguments used are similar to the ones in the previous Sections we will just describe in detail the points where differences arise. We prove in this Section that:

Proposition 25.1. For any $g \in C(X)$ there exists a unique $\varphi \in C(X)$ which solves (23.11), (23.14).

We introduce a new concept of sub and supersolutions which allows to identify the boundary condition (23.14) near the singular point $(x, v) = (0, 0)$. The rationale behind this definition is to obtain the inequality $\mathcal{L}(\varphi)(0, 0) \leq \mu_* |C_*| \mathcal{A}(\varphi)$ at the singular point. However, since we are interested in subsolutions in $L_b^\infty(X)$ some care is needed in order to define $\mathcal{L}(\varphi)(0, 0)$. Notice that the inequality $\varphi - \lambda \mathcal{L}(\varphi) \leq g$, suggests the inequality $\varphi(0, 0) - g(0, 0) \leq \lambda \mu_* |C_*| \mathcal{A}(\varphi)$.

Definition 25.2. Suppose that $g \in C(X)$. Let $\zeta = \zeta(x, v)$ a nonnegative C^∞ test function supported in $0 \leq x + |v|^3 \leq 2$, satisfying $\zeta = 1$ for $0 \leq x + |v|^3 \leq 1$. We will say that a function $\varphi \in L_b^\infty(X)$, such that the limit $\lim_{(x,v) \rightarrow (0,0)} \varphi(x, v) = \varphi(0, 0)$ exists, is a subsolution of (23.11), (23.14) if for all ψ with $\psi \geq 0$, $\psi(0, -v) = r^2 \psi(0, rv)$, $v > 0$ and having the form $\psi = \theta \zeta G_\gamma + \bar{\psi}$ where $\bar{\psi} \in C^\infty$, $\theta \geq 0$ with G_γ as in (4.8), $\gamma \in \{-\frac{2}{3}, \alpha\}$ and with ψ supported in a set contained in the ball $|(x, v)| \leq R$ for some $R > 0$ we have:

$$(25.1) \quad \int (\psi - \lambda \mathcal{L}^*(\psi)) \varphi + \frac{\theta}{\mu_*} (\varphi(0, 0) - g(0, 0)) \leq \int g \psi$$

Given $g \in C(X)$, we will say that $\varphi \in L_b^\infty(X)$ is a supersolution of (23.11), (23.13) if for all ψ with the same properties as above we have:

$$(25.2) \quad \int (\psi - \lambda \mathcal{L}^*(\psi)) \varphi + \frac{\theta}{\mu_*} (\varphi(0, 0) - g(0, 0)) \geq \int g \psi$$

The integral on the left hand side of (25.1), (25.2) is well defined in spite of the singularity of G_γ near the singular point because $\mathcal{L}^*(G_\gamma) = 0$.

We can then prove the following variation of Lemma 24.3.

Lemma 25.3. Let $g \in C(X)$ and φ be a subsolution of (23.11), (23.14) in the sense of Definition 25.2. Let:

$$(25.3) \quad \Psi(\delta) = \int_{\partial \mathcal{R}_\delta} (n_v G_\alpha \partial_v \varphi - n_v \partial_v G_\alpha \varphi + v \varphi G_\alpha n_x) ds$$

which is well defined in the sense of Traces (cf. Proposition 19.3). Moreover, the limit $\lim_{\delta \rightarrow 0^+} \Psi(\delta)$ exists and we have:

$$(25.4) \quad \lim_{\delta \rightarrow 0^+} \Psi(\delta) + \frac{1}{\lambda \mu_*} (\varphi(0, 0) - g(0, 0)) \leq 0$$

Let $\mu = -\varphi + \lambda \mathcal{L}(\varphi) + g$. Then $\mu \geq 0$ defines a Radon measure in $X \setminus \{(0, 0)\}$ satisfying:

$$(25.5) \quad \int_{\mathcal{R}_1 \setminus \{(0,0)\}} G_\alpha \mu dx dv < \infty$$

Proof. The proof is similar to the one of Lemma 24.3. The only difference is that for test functions as in Definition 25.2 we obtain the inequality:

$$\int ((\psi - \lambda \mathcal{L}^*(\psi)) \varphi - g \psi) + \frac{\theta}{\mu_*} (\varphi(0, 0) - g(0, 0)) \leq 0$$

instead of (24.5). Arguing then as in the proof of Lemma 24.3 we obtain (25.4). The rest of the argument is basically identical to the proof of Lemma 24.3 with minor changes. \square

We now remark that several of the arguments used in Section 23.3 can be adapted with minor changes to the case considered in this Section. Indeed, Lemma 24.4 holds if instead of considering subsolutions of (23.11), (23.13) in the sense of Definition 24.2 we consider subsolutions of (23.11), (23.14) in the sense of Definition 25.2. Moreover, the limit value $\varphi(0, 0)$ is the same value as the limit $\lim_{(x,v) \rightarrow (0,0)} \varphi(x, v)$ whose existence was assumed in Definition

25.2. Actually, the use of Lemma 24.4 could be avoided in the case of partially trapping Boundary Conditions, due to the existence of the limit $\lim_{(x,v) \rightarrow (0,0)} \varphi(x, v)$ in the Definition of subsolutions, however, it will be convenient to have also the corresponding version of Lemma 24.4 in order to adapt the argument yielding to the corresponding version of Lemma 24.6. The proof of a new version of Lemma 24.6 in which the subsolutions considered are subsolutions of (23.11), (23.14) in the sense of Definition 25.2 follows with minor changes, just replacing at several places inequalities like $\Psi_\varphi(0) \leq 0$ by $\Psi_\varphi(0) + \frac{1}{\lambda\mu_*}(\varphi(0, 0) - g(0, 0)) \leq 0$.

We can now adapt Lemma 24.7 as follows:

Lemma 25.4. *For any $g \in C(X)$ there exist at least one subsolution φ^{sub} and one supersolution φ^{sup} in the sense of Definition 25.2 such that:*

$$(25.6) \quad \varphi^{sub} \leq \varphi^{sup}$$

Proof. We can just take $\varphi^{sub} = -\|g\|_{L^\infty(X)}$, $\varphi^{sup} = \|g\|_{L^\infty(X)}$. The integral terms in (25.1), (25.2) can be estimated as in the proof of Lemma 24.7. Using then that $(\varphi^{sub} - g(0, 0)) \leq 0$ and $(\varphi^{sup} - g(0, 0)) \leq 0$ the result follows. \square

We can then define $\tilde{\mathcal{G}}_{sub}$ as in Definition the following subset of $L_b^\infty(X)$:

Definition 25.5. *Suppose that φ^{sub} , φ^{sup} are respectively one subsolution and one supersolution in the sense of Definition 25.2 satisfying (25.6). We then define $\tilde{\mathcal{G}}_{sub} \subset L_b^\infty(X)$ as:*

$$(25.7) \quad \tilde{\mathcal{G}}_{sub} \equiv \left\{ \varphi \in L_b^\infty(X) : \begin{array}{l} \varphi \text{ sub-solution of (23.11), (23.14)} \\ \text{in the sense of Definition 25.2} \mid \varphi^{sub} \leq \varphi \leq \varphi^{sup} \end{array} \right\}.$$

Lemmas 23.13 and 23.15 hold without any changes and the proof of Lemma 23.16 can be adapted to include this case.

Lemma 25.6. *Let $\varphi_1, \varphi_2, \dots, \varphi_L \in \tilde{\mathcal{G}}_{sub}$ with $L < \infty$ and $\tilde{\mathcal{G}}_{sub}$ as in Definition 25.2. Define:*

$$\bar{\varphi} := \max\{\varphi_1, \varphi_2, \dots, \varphi_L\}$$

Then $\bar{\varphi} \in \tilde{\mathcal{G}}_{sub}$.

Proof. It is similar to the Proof of Lemma 24.9. \square

We remark that Proposition 20.2 and Lemma 20.4 are independent of the boundary conditions imposed at the singular point $(x, v) = (0, 0)$ and therefore can be applied to subsolutions as in Definition 25.2. Lemma 23.17 can be proved for the subsolutions in the class $\tilde{\mathcal{G}}_{sub}$ given in Definition 25.2 with minor cases with respect to the case of trapping boundary conditions. We can now conclude the Proof of Proposition 25.1.

End of the Proof of Proposition 25.1. We define φ_* as in (23.28) with $\tilde{\mathcal{G}}_{sub}$ as in Definition 25.5. Arguing as in the Proof of Proposition 23.7 we obtain that $\varphi_* \in \tilde{\mathcal{G}}_{sub}$ with $\tilde{\mathcal{G}}_{sub}$ as in Definition 25.5. We define $\nu = -(\varphi_* - \lambda\mathcal{L}(\varphi_*) - g)$. Due to Definition 25.2 we have that ν is a nonnegative Radon measure. If $\nu(X \setminus \{0, 0\}) > 0$ we can argue as in the Proof of Propositions 23.7, 24.1 to derive a contradiction. Therefore $\nu(X \setminus \{0, 0\}) = 0$ and then, φ_* satisfies the problem:

$$\varphi_* - \lambda\mathcal{L}(\varphi_*) - g = 0, \quad (x, v) \in X \setminus \{0, 0\}, \quad \varphi_* \in C(X)$$

We now remark that Theorem 21.11 implies:

$$\varphi_*(x, v) = \varphi_*(0, 0) + \mathcal{A}(\varphi_*)F_\beta(x, v) + \psi_*(x, v)$$

Using Lemma 25.3 and Lemma 24.5 we obtain that $\varphi_*(0, 0) - g(0, 0) \leq \lambda\mu_*|C_*|\mathcal{A}(\varphi_*)$. Suppose that $\varphi_*(0, 0) - g(0, 0) < \lambda\mu_*|C_*|\mathcal{A}(\varphi_*)$. We can obtain a larger subsolution in the

family $\tilde{\mathcal{G}}_{sub}$ as follows. If $\varphi_*(0,0) = \varphi^{sup}$ we would obtain that $\varphi_{**} = \max \{ \varphi_*, \varphi_*(0,0) + \delta + CS(x,v) + \mathcal{A}(\varphi_*) \}$ with $S(x,v)$ as in Lemma 21.1 is a subsolution of (23.11), (23.14) in a neighbourhood of the singular point satisfying $\varphi_{**} > \varphi_*$, for a constant C depending only of g if δ is sufficiently small, since S is bounded by $\left(x^{\frac{2}{3}} + v^2\right)$ and therefore $\mathcal{A}(\varphi_*) F_\beta$ gives a larger contribution for (x,v) close to $(0,0)$. However this contradicts the definition of φ_* , whence $\varphi_*(0,0) - g(0,0) \leq \lambda \mu_* |C_*| \mathcal{A}(\varphi_*)$. Then $\mathcal{L}(\varphi_*)(0,0) = \mu_* |C_*| \mathcal{A}(\varphi_*)$, whence $\varphi_* \in \mathcal{D}(\Omega_{pt,sub})$ and the result follows. \square

26. OPERATOR Ω_{sup} . SOLVABILITY OF (23.11), (23.15).

In the supercritical case $r > r_c$ we cannot impose a boundary condition at the singular point. However, we can use also Perron's method in this case. We will include a reference to (23.15) in order to make clear that we refer to the supercritical case. We will prove the following result:

Proposition 26.1. *For any $g \in C(X)$ there exists a unique $\varphi \in C(X)$ which solves (23.11), (23.15).*

The concept of sub and supersolutions which we will use in this case is the following:

Definition 26.2. *Suppose that $g \in C(X)$. We will say that a function $\varphi \in L_b^\infty(X)$, is a subsolution of (23.11), (23.15) if for all $\psi \in C^\infty(\bar{U})$ with $\psi \geq 0$, $\psi(0, -v) = r^2 \psi(0, rv)$, $v > 0$ and with ψ supported in a set contained in the ball $|(x,v)| \leq R$ for some $R > 0$ we have:*

$$(26.1) \quad \int (\psi - \lambda \mathcal{L}^*(\psi)) \varphi \leq \int g \psi$$

Given $g \in C(X)$, we will say that $\varphi \in L_b^\infty(X)$ is a supersolution of (23.11), (23.15) if for all ψ with the same properties as above we have:

$$(26.2) \quad \int (\psi - \lambda \mathcal{L}^*(\psi)) \varphi \geq \int g \psi$$

Remark 26.3. *Notice that the assumption $\psi \in C^\infty(\bar{U})$ does not imply any condition in the values of the derivatives of ψ at $\{x=0\}$, except those which follow from the compatibility condition $\psi(0, -v) = r^2 \psi(0, rv)$, $v > 0$.*

We can now construct easily bounded sub and supersolutions in the sense of Definition 26.2:

Lemma 26.4. *For any $g \in C(X)$ there exist at least one subsolution φ^{sub} and one supersolution φ^{sup} in the sense of Definition 26.2 such that:*

$$(26.3) \quad \varphi^{sub} \leq \varphi^{sup}$$

Proof. We can just take $\varphi^{sub} = -\|g\|_{L^\infty(X)}$, $\varphi^{sup} = \|g\|_{L^\infty(X)}$. The proof can be made then by means of a small adaptation of the one of Lemma 24.7. \square

We can then define $\tilde{\mathcal{G}}_{sub}$ as in Definition the following subset of $L_b^\infty(X)$:

Definition 26.5. *Suppose that φ^{sub} , φ^{sup} are respectively one subsolution and one supersolution in the sense of Definition 26.2 satisfying (26.3). We then define $\tilde{\mathcal{G}}_{sub} \subset L_b^\infty(X)$ as:*

$$(26.4) \quad \tilde{\mathcal{G}}_{sub} \equiv \left\{ \varphi \in L_b^\infty(X) : \begin{array}{l} \varphi \text{ sub-solution of (23.11), (23.15)} \\ \text{in the sense of Definition 26.2} \mid \varphi^{sub} \leq \varphi \leq \varphi^{sup} \end{array} \right\}.$$

In this case we do not need to study the subsolutions near the singular point with the same level of detail as in the subcritical cases. However, we will need to be able to solve the Dirichlet problem with boundary values in the admissible boundaries for domains containing the singular point. The following result generalizes Proposition 20.2 to the case of domains containing the singular point if $r > r_c$:

Proposition 26.6. *Suppose that $r > r_c$ and let Λ_R as in (20.23) for some $R > 0$. Let us assume also that $g \in C(\overline{\Lambda_R})$, and $h \in L^\infty(\partial_a \Lambda_R)$ where $\partial_a \Lambda_R$ is the corresponding admissible boundary of Λ_R . Then, there exists a unique classical solution of the problem (20.1), (20.2) where the boundary condition (20.2) is achieved in the sense of trace defined in Proposition 19.3.*

Proof. The existence of solutions can be obtained considering the sequence of solutions of the problems (20.1), (20.2) in the domains $\Lambda_R \setminus \mathcal{R}_\delta$ with $\delta > 0$ small and boundary conditions $\varphi = h$ on $\partial_a \Lambda_R$ and, say, $\varphi = 0$ on $\partial_a(\Lambda_R \setminus \mathcal{R}_\delta) \cap \partial \mathcal{R}_\delta$. The solutions of these problems, which will be denoted as φ_δ are uniformly bounded by $\|g\|_{L^\infty} + \|h\|_{L^\infty}$ due to the maximum principle. Moreover, the derivatives of the functions φ_δ are bounded in $W_{loc}^{1,p}$ due to Theorem 12.4. Therefore, a standard compactness argument allows to find a subsequence φ_{δ_n} converging uniformly to a weak solution of (20.1), (20.2).

In order to prove uniqueness we consider the difference ψ of two bounded solutions of (20.1), (20.2) in Λ_R satisfies $(\psi - \lambda \mathcal{L}\psi) = 0$ in Λ_R , $\psi = 0$ on $\partial_a \Xi$. We can construct now a positive supersolution with the form εF_β , $\varepsilon > 0$, since $\varepsilon(F_\beta - \lambda \mathcal{L}F_\beta) = \varepsilon F_\beta > 0$. Since ψ is bounded, and $\beta < 0$ for $r > r_c$ we obtain that $|\psi| \leq \varepsilon F_\beta$ if we choose δ small enough (depending on ε). We can then use a comparison argument in the domain $\Lambda_R \setminus \mathcal{R}_\delta$ whence $|\psi| \leq \varepsilon F_\beta$ in Λ_R for $\varepsilon > 0$ arbitrary. Taking the limit $\varepsilon \rightarrow 0$ we obtain $\psi = 0$ and the uniqueness result follows. \square

End of the Proof of Proposition 26.1. We define φ_* as in (23.28) with $\tilde{\mathcal{G}}_{sub}$ as in Definition 26.5. Arguing as in the Proof of Propositions 23.7, 24.1, 25.1 we obtain that $\varphi_* \in \tilde{\mathcal{G}}_{sub}$. We define $\nu = -(\varphi_* - \lambda \mathcal{L}(\varphi_*) - g)$. Due to Definition 26.5 we have that ν is a nonnegative Radon measure. If $\nu \neq 0$ we consider a domain Ξ that is, either one of the admissible domains Ξ in Definition 12.1 or one of the domains Λ_R containing the singular point as in the statement of Proposition 26.6. We can then argue as in the Proof of Propositions 23.7, 24.1, 25.1 to derive a contradiction. Therefore $\nu = 0$ and then, $\varphi_* \in C(X)$ gives the desired solution of the problem (23.11), (23.15). \square

27. SOLVABILITY OF THE ADJOINT PROBLEMS (16.3)-(16.6), (16.7)-(16.10), (16.11)-(16.15), (16.16)-(16.19).

We can now prove the solvability of the adjoint problems described in Section 16 of Section 3.

Theorem 27.1. *Suppose that Ω_σ is one of the operators $\Omega_{t,sub}$, $\Omega_{nt,sub}$, $\Omega_{pt,sub}$, Ω_{sup} defined in (13.2)-(13.3), (13.4)-(13.5), (13.6)-(13.7), (13.8)-(13.9) respectively. We can define Markov semigroups $S_\sigma(t)$ having as generator the operator Ω_σ . For any $\varphi \in \mathcal{D}(\Omega_\sigma)$ we can define a function $u \in C^1([0, \infty) : C(X))$ such that:*

$$\partial_t u = \Omega_\sigma u \quad , \quad t \in [0, \infty) \quad , \quad u(t, \cdot) \in \mathcal{D}(\Omega_\sigma) \quad \text{if } t \geq 0 \quad , \quad u(0, \cdot) = \varphi$$

Moreover, u is a classical solution of the equation $\partial_t u(x, v, t) = \mathcal{L}u(x, v, t)$ for $(x, v) \neq (0, 0)$, $t > 0$.

Remark 27.2. *Notice that the compatibility condition (11.1) is a consequence of the fact that $u(\cdot, t) \in C(X)$ for any $t \geq 0$.*

Proof. Theorem 27.1 is a consequence of the Hille-Yosida Theorem (cf. Theorem 18.7), as well as Proposition 22.1, Proposition 23.4 and Propositions 23.7, 24.1, 25.1 and 26.1. \square

28. WEAK SOLUTIONS FOR THE ORIGINAL PROBLEM.

Our next goal is to define suitable measured valued solutions of the problem (1.3)-(1.5) by means of the adjoint problems (16.16)-(16.19), (16.3)-(16.6), (16.7)-(16.10), (16.11)-(16.15). To this end, we argue by duality. We will use the index σ to denote each of the four cases considered in Section 13 of Section 8, namely, in the case of subcritical values of r , we can use trapping, nontrapping or partially trapping boundary conditions. We will consider also the supercritical case. The following definition will be used to define measured valued solutions of the problem (1.3)-(1.5) with all the boundary conditions considered above.

Definition 28.1. Given $P_0 \in \mathcal{M}_+(X)$ we define a measured valued function $P_\sigma \in C([0, \infty) : \mathcal{M}_+(X))$ by means of:

$$(28.1) \quad \int \varphi(dP_\sigma(\cdot, t)) = \int (S_\sigma(t) \varphi)(dP_0) \quad , \quad t \geq 0$$

for any $\varphi \in C([0, \infty) : C(X))$

Remark 28.2. Notice that the notation in (28.1) must be understood as follows. Let $\psi(\cdot, t) = S_\sigma(t) \varphi$. Then, the right-hand side of (28.1) is equivalent to $\int \psi(\cdot, t) dP_0$. The left-hand side of (28.1) is just $\int \varphi(\cdot) dP_\sigma(t)$.

It is convenient to write in detail the weak formulation of (1.3)-(1.5) satisfied by each of the measures P_σ .

Definition 28.3. Suppose that $0 < r < r_c$ and $P_0 \in \mathcal{M}_+(X)$. We will say that $P \in C([0, \infty) : \mathcal{M}_+(X))$ is a weak solution of (1.3)-(1.5) with trapping boundary conditions if for any $T \in [0, \infty)$ and φ such that, for any time $t \in [0, T]$, $\varphi(\cdot, t), \mathcal{L}\varphi(\cdot, t), \varphi_t(\cdot, t) \in C(X)$, $\varphi(\cdot, t)$ satisfies (13.1) and such that $\mathcal{L}\varphi(0, 0, t) = 0$ for any $t \geq 0$, the following identity holds:

$$(28.2) \quad \int_0^T \int_{X \setminus \{(0,0)\}} [\varphi_t(dP(\cdot, t), t) + \mathcal{L}\varphi(dP(\cdot, t), t)] + \int_X \varphi(dP_0(\cdot), 0) - \int_X \varphi(dP(\cdot, T), T) = 0,$$

where \mathcal{L} is as in (11.3).

Definition 28.4. Suppose that $0 < r < r_c$ and $P_0 \in \mathcal{M}_+(X)$. We will say that $P \in C([0, \infty) : \mathcal{M}_+(X))$ is a weak solution of (1.3)-(1.5) with nontrapping boundary conditions if for any $T \in [0, \infty)$ and φ such that, for any time $t \in [0, T]$, $\varphi(\cdot, t), \mathcal{L}\varphi(\cdot, t), \varphi_t(\cdot, t) \in C(X)$, $\varphi(\cdot, t)$ satisfies (13.1) and $\mathcal{A}(\varphi)(\cdot, t) = 0$ the (28.2) identity holds, where \mathcal{L} is as in (11.3).

Definition 28.5. Suppose that $0 < r < r_c$ and $P_0 \in \mathcal{M}_+(X)$. We will say that $P \in C([0, \infty) : \mathcal{M}_+(X))$ is a weak solution of (1.3)-(1.5) with partially trapping boundary conditions if for any $T \in [0, \infty)$ and φ such that, for any time $t \in [0, T]$, $\varphi(\cdot, t), \mathcal{L}\varphi(\cdot, t), \varphi_t(\cdot, t) \in C(X)$, $\varphi(\cdot, t)$ satisfies (13.1) and $\mathcal{L}\varphi(0, 0, t) = \mu_* |C_*| \mathcal{A}(\varphi)$, the identity (28.2) holds, where \mathcal{L} is as in (11.3).

Definition 28.6. Suppose that $r > r_c$ and $P_0 \in \mathcal{M}_+(X)$. We will say that $P \in C([0, \infty) : \mathcal{M}_+(X))$ is a weak solution of (1.3)-(1.5) for supercritical boundary conditions if for any $T \in [0, \infty)$ and $\varphi \in C_c^2([0, T] \times X)$ the identity (28.2) holds, where \mathcal{L} is as in (11.3).

Our next goal is to prove the following result:

Proposition 28.7. *Suppose that we define measure valued functions*

$$P_{t,\text{sub}}, P_{nt,\text{sub}}, P_{pt,\text{sub}}, P_{\text{sup}} \in C([0, \infty) : \mathcal{M}_+(X))$$

as in Definition 28.1 and initial datum P_0 . Then, they are weak solutions of (1.3)-(1.5) with trapping, nontrapping, partially trapping and supercritical boundary conditions respectively. Moreover, the functions $P_{t,\text{sub}}, P_{nt,\text{sub}}, P_{pt,\text{sub}}, P_{\text{sup}}$ are the unique solutions of (1.3)-(1.5) with trapping, nontrapping, partially trapping and supercritical boundary conditions in the sense of Definitions 28.3, 28.4, 28.5 and 28.6 respectively.

Proof. We take a smooth test function φ depending on the variables (x, v, t) . We will assume that the function $\varphi(\cdot, t) \in \mathcal{D}(\Omega_\sigma)$ for any $t \in [0, T]$. The definition of the Probability measure P yields:

$$\int \varphi(dP_\sigma(\cdot, t), \bar{t}) = \int (S_\sigma(t) \varphi)(dP_0, \bar{t})$$

for any fixed $\bar{t} > 0, t > 0$. In particular, taking for \bar{t} the values $(t + h)$ with h small, we obtain:

$$\int \varphi(dP_\sigma(\cdot, t), t + h) = \int (S_\sigma(t) \varphi)(dP_0, t + h)$$

Differentiating this equation with respect to h and taking $h = 0$, it then follows that:

$$(28.3) \quad \int \varphi_t(dP_\sigma(\cdot, t), t) = \int (S_\sigma(t) \varphi_t)(dP_0, t)$$

We now write $S_\sigma(t) \varphi_t(\cdot, t) = \frac{d}{dt}(S_\sigma(t) \varphi(\cdot, t)) - \partial_t(S_\sigma(t)) \varphi(\cdot, t)$. Integrating then (28.3) in the time interval $(0, T)$ it then follows that:

$$\begin{aligned} \int_0^T \int \varphi_t(dP_\sigma(\cdot, t), t) dt &= \int_0^T \int \left[\frac{d}{dt}(S_\sigma(t) \varphi(dP_0, t)) - \partial_t(S_\sigma(t)) \varphi(dP_0, t) \right] dt \\ &= \int_0^T \left[\frac{d}{dt} \left(\int (S_\sigma(t) \varphi(dP_0, t)) \right) - \int \partial_t(S_\sigma(t)) \varphi(dP_0, t) \right] dt \\ &= \int S_\sigma(T) \varphi(dP_0, T) - \int (S_\sigma(0) \varphi(dP_0, 0)) \\ &\quad - \int_0^T \int \partial_t(S_\sigma(t)) \varphi(dP_0, t) dt \\ &= \int \varphi(dP_\sigma(\cdot, T), T) - \int \varphi(dP_0, 0) - \int_0^T \int \Omega_\sigma S_\sigma(t) \varphi(dP_0, t) dt \\ &= \int \varphi(dP_\sigma(\cdot, T), T) - \int \varphi(dP_0, 0) - \int_0^T \int \Omega_\sigma \varphi(dP_\sigma(\cdot, t), t) dt \end{aligned}$$

where we have used that for any function $\psi \in \mathcal{D}(\Omega_\sigma)$ we have $\partial_t(S_\sigma(t)) \psi = \Omega_\sigma S_\sigma(t) \psi$, whence:

$$\begin{aligned} &\int \varphi(dP_\sigma(\cdot, T), T) - \int \varphi(dP_0, 0) \\ &= \int_0^T \int [\varphi_t(dP_\sigma(\cdot, t), t) + \Omega_\sigma \varphi(dP_\sigma(\cdot, t), t)] dt \end{aligned}$$

Using that the action of the operators Ω_σ in their respective domains is given by \mathcal{L} we obtain the existence result.

Uniqueness can be proved using the solvability of the adjoint problems obtained in Theorem 27.1. Suppose that P_1, P_2 are two solutions of (1.3)-(1.5) with trapping boundary conditions in the sense of Definition 28.3. Then $P = P_1 - P_2$ satisfies (28.2) with $P_0 = 0$ for any function φ with the regularity requested in the Definition 28.3. Suppose that P

does not vanish identically. Then, for some $T > 0$ there exists a function $\tilde{\psi} \in C(X)$ such that $\int_X \tilde{\psi}(dP(\cdot, T)) \neq 0$. Since $\mathcal{D}(\Omega_t)$ is dense in $C(X)$ there exists $\psi \in \mathcal{D}(\Omega_t)$ such that $\int_X \psi(dP(\cdot, T)) \neq 0$. Theorem 27.1 implies the existence of a function $\varphi \in C^1([0, T]; C(X))$ satisfying $\varphi_t(\cdot, t) + \mathcal{L}\varphi(\cdot, t) = 0$ in X , for $t \in [0, T]$ and $\mathcal{L}\varphi(0, 0, t) = 0$ for any $t \in [0, T]$, with the initial condition $\varphi(\cdot, T) = \psi(\cdot)$. This function satisfies the regularity conditions required in Definition 28.3. It then follows from (28.2) that $\int_X \psi(dP(\cdot, T)) = 0$, but this gives a contradiction, whence the uniqueness result follows for trapping boundary conditions. The proof in the case of non-trapping, partially trapping and supercritical boundary conditions is similar and the details will be omitted. \square

We can also prove the following result.

Theorem 28.8. *Suppose that we define measures $P_{t,sub}, P_{nt,sub}, P_{pt,sub}, P_{sup} \in C([0, \infty) : \mathcal{M}_+(X))$ as in Definition 28.1 and initial datum P_0 . Then, these measures can be decomposed as:*

$$(28.4) \quad dP_\sigma(\cdot, t) = m_\sigma(t) \delta_{(0,0)} + p_\sigma(x, v, t) dx dv, \quad t \geq 0$$

where for each σ we have $p_\sigma(\cdot, t) \in L^1(X)$ and $m_\sigma(t) \geq 0$. If $m_\sigma(0) = 0$ we have also $m_{nt,sub}(t) = m_{sup}(t) = 0$ for any $t \geq 0$. Moreover, if P_0 is not identically zero, we have also $m_{t,sub}(t) > 0$, $m_{pt,sub}(t) > 0$ for any $t > 0$.

The functions $p_\sigma(\cdot, t)$ are infinitely differentiable for $(x, v) \neq (0, 0)$ and they satisfy (1.4).

Proof. Using in (28.2) the test function $\varphi = 1$ we obtain that, in the case of the four considered operators:

$$\int_X dP_\sigma(\cdot, t) = \int_X dP_0, \quad t \geq 0$$

We define $m_\sigma(t) = \int_{\{(0,0)\}} dP_\sigma(\cdot, t)$. We define also the measures $dP_\sigma(\cdot, t) - m_\sigma(t) \delta_{(0,0)}$. Using Proposition 28.7 it then follows that this family of measures solves (1.3), (1.4) outside the singular point $(x, v) = (0, 0)$. Using classical interior hypoellipticity results we obtain the representation (28.4) where $p_\sigma \in C^\infty$ in the set $\{x > 0\}$. On the other hand, we can prove that $p_\sigma \in C^\infty$ for $x = 0$, $v < 0$ using the fact that the characteristic curves move away from the domain $\{x > 0\}$ in that region. Actually, it is possible to argue basically as in the proof of boundary regularity derived in [31]. More precisely, introducing a cutoff function supported in the region $\{v < 0\}$ which takes the value 1 in the region where the regularity of p can be obtained, we would obtain a new function \tilde{p} solving the same Fokker-Planck equation, but with some source terms containing the derivative p_v and the function p . The arguments in [31] then allow to prove that $p \in C^\infty\{x \geq 0, v < 0\}$. Using then (1.4) we would obtain also regularity in $\{x = 0, v > 0\}$. Uniform regularity in compact sets of $\{x = 0, v > 0\}$ would then follow as in [31].

It then follows, since $dP_\sigma(\cdot, t)$ is a Radon measure, and therefore, outer regular that $p_\sigma \in L^\infty((0, T) : L^1(X))$.

In order to obtain the stated results for the masses $m_\sigma(t)$ we need to study the asymptotic behaviour of some solutions of the adjoint problems with initial data approaching the characteristic function of the singular point. The specific form of the solutions under consideration depends of the specific boundary conditions used.

In the cases of trapping and partially trapping boundary conditions we construct some particular test functions with the form:

$$\varphi(x, v, t) = \Phi\left(\frac{x}{(\bar{t} - t)^{\frac{3}{2}}}, \frac{v}{(\bar{t} - t)^{\frac{1}{2}}}\right) = \Phi(\xi, \eta)$$

where Φ satisfies:

$$-\frac{3}{2}\xi\partial_\xi\Phi - \frac{1}{2}\eta\partial_\eta\Phi \leq \partial_{\eta\eta}\Phi + \eta\partial_\xi\Phi$$

in the sense of measures. Notice that this is equivalent to obtaining the inequalities $\varphi_t(\cdot, t) + \mathcal{L}\varphi(\cdot, t) \geq 0$. In the case of trapping boundary conditions we look for functions with the form:

$$\Phi(\xi, \eta) = a - F_\beta(\xi, \eta) - W(\xi, \eta)$$

where we choose W satisfying:

$$\partial_{\eta\eta}W + \eta\partial_\xi W \leq C_0(\xi|\partial_\xi F_\beta| + |\eta||\partial_\eta F_\beta|)$$

and the function W satisfies $|W(\xi, \eta)| \leq C(|\eta|^2 + |\xi|^{\frac{2}{3}})F_\beta$. The existence of the function W and the previous estimate can be proved as Lemma 21.1. Choosing $a > 0$ sufficiently small we obtain that $|W(\xi, \eta)| < F_\beta(\xi, \eta)$ in the neighbourhood of the singular point where $\Phi > 0$. Moreover, by construction we obtain $\mathcal{L}(\Phi)(0, 0) = 0$ due to the fact that $\mathcal{L}(F_\beta) = 0$. We then define a test function $\tilde{\Phi}$ as:

$$\tilde{\Phi} = \frac{1}{a} \max\{\Phi, 0\}$$

Then $\tilde{\Phi}(0, 0) = 1$. Using the duality formula (28.1) we then obtain:

$$(28.5) \quad \int_{\mathcal{R}_\delta} P_\sigma(dx dv, \bar{t}) \geq \int_{\mathcal{R}_\rho} P_\sigma\left(dx dv, \frac{\bar{t}}{2}\right), \quad \bar{t} > 0$$

where $\rho > 0$ is small but it can be chosen independently of δ , and $\delta > 0$ can be taken arbitrarily small. The integral on the right can be estimated uniformly from below, because P is strictly positive in a set with the form $\mathcal{R}_\rho \setminus \mathcal{R}_{\frac{\rho}{2}} \cap \{x > 0\}$ for any fixed $\bar{t} > 0$ due to the strong maximum principle. This type of strong maximum principle arguments in interior domains have been used also in [31]. Taking the limit $\delta \rightarrow 0$ we obtain $\int_{\{(0,0)\}} P_\sigma(dx dv, \bar{t}) > 0$ for any $\bar{t} > 0$. This gives $m_{t,sub}(\bar{t}) > 0$.

In the case of partially trapping boundary conditions we will look for test functions satisfying $\varphi_t(\cdot, t) + \mathcal{L}\varphi(\cdot, t) \geq 0$ in the sense of measures, constructed by means of the auxiliary function:

$$\varphi(x, v, t) = a - F_\beta(\xi, \eta) - (\bar{t} - t)^{1-\frac{3\beta}{2}} W(\xi, \eta)$$

where $(\xi, \eta) = \left(\frac{x}{(\bar{t}-t)^{\frac{3}{2}}}, \frac{v}{(\bar{t}-t)^{\frac{1}{2}}}\right)$. Notice that for this function we have $\mathcal{A}(\varphi) = -(\bar{t} - t)^{-\frac{3\beta}{2}}$ and $\mathcal{L}(\varphi)(0, 0) = -(\bar{t} - t)^{-\frac{3\beta}{2}} \mathcal{L}(W)(0, 0)$. Moreover, since $\beta < \frac{1}{6}$ we have $\left(1 - \frac{3\beta}{2}\right) > 0$. The boundary condition for φ then becomes:

$$(28.6) \quad \mathcal{L}(W)(0, 0) = \mu_* |C_*|$$

Writing the equation we obtain that φ satisfies $\varphi_t(\cdot, t) + \mathcal{L}\varphi(\cdot, t) \geq 0$ if:

$$\mathcal{L}(W) \leq -C(|W| + F_\beta)$$

for a suitable constant $C > 0$ and where we have to impose the boundary condition (28.6). The resulting function W is quadratic near the singular point. Therefore, for small $|(\xi, \eta)|$, and given that $\left(1 - \frac{3\beta}{2}\right) > 0$ it follows that $F_\beta(\xi, \eta) \gg (\bar{t} - t)^{1-\frac{3\beta}{2}} W(\xi, \eta)$. Then, if we choose $a > 0$ small enough, and we define:

$$\tilde{\varphi} = \max(\varphi, 0)$$

it follows that $\tilde{\varphi}$ is a test function, whose support expands in a self-similar way and that it can be used as in the case of trapping boundary conditions to prove that $m_{pt,sub}(t) > 0$.

Notice that these test functions yield that $\int_{\{0\}} dP_\sigma(\cdot, \bar{t}) > 0$ for any $\bar{t} > 0$, using (28.5), as well as the strong maximum principle which guarantees that $P\left(\cdot, \frac{\bar{t}}{2}\right) > 0$ in a neighbourhood of the origin in the set $\{x > 0\}$.

In order to prove that $m_{nt,sub}(t) = m_{sup}(t) = 0$ we need to obtain test functions satisfying $\varphi_t(\cdot, t) + \mathcal{L}\varphi(\cdot, t) \leq 0$ as well as $\mathcal{A}(\varphi) = 0$ and taking initial values "close" to the characteristic function of the singular point and approaching to zero very fast.

It is natural to look for test functions which at least in some regions will have the form:

$$(28.7) \quad \varphi(x, v, t) = 2\delta^\gamma [(\bar{t} - t) + \delta]^{-\gamma} \Phi\left(\frac{x}{(\bar{t} - t + \delta)^{\frac{3}{2}}}, \frac{v}{(\bar{t} - t + \delta)^{\frac{1}{2}}}\right)$$

for $\delta > 0$ fixed. Plugging this in the equation $\varphi_t + \mathcal{L}(\varphi) = 0$, and looking for functions satisfying $\varphi_t(\cdot, t) + \mathcal{L}\varphi(\cdot, t) \leq 0$ we obtain the inequality:

$$(28.8) \quad \gamma\Phi + \frac{3}{2}\xi\Phi_\xi + \frac{1}{2}\eta\Phi_\eta + \mathcal{L}(\Phi) \leq 0$$

We need to obtain a function Φ satisfying (28.8) in the sense of measures. We consider first the case $r < r_c$ with non-trapping boundary conditions. To this end we consider a function Φ smooth outside the origin, and homogeneous function, say Φ_1 satisfying $\frac{3}{2}\xi\Phi_\xi + \frac{1}{2}\eta\Phi_\eta = -a\Phi$ with $a > 0$ fixed, independent of γ . Notice that we can assume that Φ_1 tends to infinity as (ξ, η) approaches the singular point. Then, the inequality (28.8) holds for $|\xi| + |\eta|$ large, because the contribution of the term $\mathcal{L}(\Phi)$ becomes smaller than $-a\Phi$ for large $|\xi| + |\eta|$ and $\gamma\Phi$ gives a small contribution if γ is small. On the other hand, in order to obtain a bounded function for $|\xi| + |\eta|$ of order one we construct a function Φ_2 which takes the value 1 at the origin and is corrected by means of one function W satisfying $C\gamma + \mathcal{L}(W) \leq 0$. This function can be constructed satisfying (28.8) in any bounded region, as large as desired, if γ is chosen small, because then the contribution of the corrective term is small. Taking the minimum of Φ_1 , Φ_2 we obtain the desired test function satisfying (28.8) in the sense of measures and the result follows.

In the supercritical case we can obtain a functions satisfying (28.8) as follows. The function F_β satisfies $\mathcal{L}(F_\beta) = 0$ and in the supercritical case $\beta < 0$. Then the contribution of the terms $\frac{3}{2}\xi\Phi_\xi + \frac{1}{2}\eta\Phi_\eta$ is negative close to the origin. Moreover, if we choose γ sufficiently small we obtain that $\gamma\Phi + \frac{3}{2}\xi\Phi_\xi + \frac{1}{2}\eta\Phi_\eta$ is negative. In order to obtain a bounded test function we construct an auxiliary function, taking the value $\Phi(0, 0) = 1$ and corrected by a term W satisfying $C\gamma + \mathcal{L}(W) \leq 0$ in the usual manner. Taking then the minimum between both functions we obtain a new bounded test function satisfying (28.8) in the sense of measures.

Notice that in both cases (supercritical case and subcritical with nontrapping boundary conditions), the function (28.7) is larger than one in a neighbourhood of the singular point for short times, but it decreases to values of order δ in times of order, say $\sqrt{\delta}$ if δ is small. This implies, taking the limit $\delta \rightarrow 0$ that the mass at the singular point at any time $\bar{t} > 0$ is zero, using the duality formula $\int_{\{(0,0)\}} dP_\sigma(\cdot, \bar{t}) \varphi(\bar{t}, \cdot) \leq \int_X dP_\sigma\left(\cdot, \frac{\bar{t}}{2}\right) \varphi\left(\cdot, \frac{\bar{t}}{2}\right) \leq \varepsilon_0$, where ε_0 can be made arbitrarily small if δ is small. \square

29. STATIONARY SOLUTIONS IN A STRIP.

In this Section we recall several of the stationary solutions obtained for the problem (1.3)-(1.5) in the strip $0 < x < 1$ in the physics literature and we discuss the specific set of boundary conditions for which they are solutions. We notice that we can define the adjoint problems and extend the solutions of (1.3)-(1.5) as in the Definitions 28.3, 28.4, 28.5, 28.6 using the fact that the asymptotic behaviour of the functions φ is local near the singular points $(x, v) = (0, 0)$ and $(x, v) = (1, 0)$.

We can then construct different stationary solutions for the problem (1.3)-(1.5) with trapping, nontrapping, partially trapping and supercritical conditions respectively.

In the case of trapping boundary conditions we can construct a family of solutions depending on two parameters and given by:

$$(29.1) \quad P(x, v) = m_1 \delta_{(0,0)} + m_2 \delta_{(1,0)} \quad , \quad m_1 \geq 0, \quad m_2 \geq 0$$

if $0 < r < r_c$.

A class of stationary solutions of (1.3)-(1.5) has been obtained in [7], [8]. This solution is obtained by means of the solution of an integral equation. We just indicate the main properties of such solutions which are relevant here. The solution obtained in [7], [8] will be denoted as $F(x, v)$ and it satisfies:

$$(29.2) \quad v \partial_x F = \partial_{vv} F \quad , \quad (x, v) \in \{0 < x < 1, \quad v \in \mathbb{R}\}$$

$$(29.3) \quad F(x, v) = F(1 - x, -v)$$

$$(29.4) \quad F(0, -v) = r^2 F(0, rv) \quad , \quad v > 0$$

as well as the asymptotics:

$$(29.5) \quad F(x, v) \sim G_\alpha(x, v) + o\left(x^{\frac{2}{3}} + v^2\right) \quad \text{as } (x, v) \rightarrow (0, 0)$$

and an analogous asymptotics near the singular point $(x, v) = (1, 0)$ that can be obtained using the symmetry condition (29.3). This asymptotics (29.5) corresponds to the nontrapping boundary conditions. Therefore, we obtain a family of stationary solutions of (1.3)-(1.5) with nontrapping boundary conditions if given by:

$$(29.6) \quad P(x, v) = C_0 F(x, v) \quad , \quad (x, v) \in \{0 \leq x \leq 1, \quad v \in \mathbb{R}\} \quad , \quad C_0 \geq 0$$

if $0 < r < r_c$. Moreover, the solutions (29.6) yield also stationary solutions of (1.3)-(1.5) for $r > r_c$.

We can obtain stationary solutions in the strip $\{0 < x < 1, \quad v \in \mathbb{R}\}$ with partially trapping boundary conditions by means of:

$$(29.7) \quad P(x, v) = C_0 \left[F(x, v) + \frac{1}{\mu_*} (\delta_{(0,0)} + \delta_{(1,0)}) \right] \quad , \quad C_0 \geq 0$$

with $0 < r < r_c$. Notice that these solutions satisfy the boundary condition (5.10).

Using arguments analogous to those in Section 8 we might show that the measures (29.1), (29.6), (29.7) are stationary solutions of (1.3)-(1.5) with trapping, nontrapping and partially trapping boundary conditions in the sense of the definitions 28.3, 28.4, 28.5 respectively. On the other hand (29.6) is a solution of (1.3)-(1.5) in the supercritical case $r > r_c$ in the sense of the definition 28.6.

30. THE PROBLEM WITH ABSORBING BOUNDARY CONDITIONS.

The Fokker-Planck equation with absorbing boundary conditions in one space dimension has been recently considered in [31]. We summarize some of the most relevant results obtained there.

We have proved in that paper that the solutions of (1.1) in the strip $\{0 < x < 1, v \in \mathbb{R}\}$ with absorbing boundary conditions, i.e. $P(0, v, t) = 0$ if $v > 0$, $P(1, v, t) = 0$ if $v < 0$ are Hölder continuous in the closed strip $\{0 \leq x \leq 1, v \in \mathbb{R}\}$. Moreover, the solutions are C^∞ in the interior of this strip as well as at the boundaries $\{x = 0, v \neq 0\}$ and $\{x = 1, v \neq 0\}$. The only points where the solutions are not C^∞ are the singular points $(x, v) = (0, 0)$ and $(x, v) = (1, 0)$. In those points P has a singularity with self-similar form and the solutions of (1.1) are at most Hölder continuous there. Actually, it has been proved in [31] that the solutions of (1.1) in the strip $\{0 < x < 1, v \in \mathbb{R}\}$ with absorbing boundary conditions are

Hölder continuous in a neighbourhood of the singular points, and the optimal hölderianity exponent has been obtained there. These regularity properties were extended in [30] to the case of the three dimensional half spaces and in [29] to the case of bounded domains with curved boundaries, where the convexity was relaxed for the hypoellipticity up to the boundary unlike the Vlasov and Boltzmann equations.

The long time asymptotics of the solutions of those problems has been also considered in [31], where it has been proved that the mass escapes from the strip

$$\{0 < x < 1, v \in \mathbb{R}\}$$

and as a consequence the total mass of P in the strip $\{0 < x < 1, v \in \mathbb{R}\}$ decreases exponentially. The hypoellipticity results (up to the boundaries) obtained in [31] allow us to prove that the solutions of (1.1) decrease exponentially as well as their derivatives in any compact set which does not include the singular points $(x, v) = (0, 0)$ and $(x, v) = (1, 0)$.

Probabilistic problems closely related to (1.1) have been considered in the physics literature. As discussed in the Introduction, the probability distribution for the velocity with which a Brownian particle exits from the domain $\{x > 0\}$ is computed in [45].

31. OPEN PROBLEMS

We have shown that inelastic conditions like in (1.4) allow us to impose different classes of boundary conditions at the singular set in one-dimensional problems. We have seen that, assuming that $r < r_c$ it is possible to impose trapping, nontrapping and partially trapping boundary conditions in a natural manner. It would be relevant to ascertain what is exactly the situation in higher dimensional problems, since in those cases, if the domain W where the problem is solved is contained in \mathbb{R}^d , it turns out that the singular set is a $(2d - 2)$ -dimensional set. The detailed structure of singularities and the specific form of the admissible boundary conditions in the singular set are an interesting mathematical question. We have not studied in detail in this paper the case of the critical exponent $r = r_c$, in which logarithmic corrections to the solutions might be expected to appear near the singular points. The results in this paper do not cover the case $r > 1$, for which uniqueness of solutions might be expected, although on the other hand the physical meaning of this problem is "a priori" unclear. Notice that in the limit case $r \rightarrow \infty$ we might expect convergence of the solutions in bounded regions to the solutions of the problem with absorbing boundary conditions which was studied in [31]. We also remark that in the case $r = 1$ it is possible to obtain solutions of (1.3)-(1.5) in a simple form using the fact that the solutions of the problem in the whole space with initial data satisfying $P_0(x, v) = P_0(-x, -v)$ solve the problem (1.3)-(1.5) in $x > 0$ with $r = 1$.

An issue that has been considered in [31] is the long time asymptotics of the solutions in a strip with absorbing boundary conditions. It turns out that in that case the amount of mass left in the strip is bounded by an exponentially decreasing function. In the situation considered in this paper, the expected behaviour is the convergence to one stationary solution. The same issue can be formulated in unbounded domains. In the case of half-planes one can expect self-similar behaviours for the solutions.

It might be interesting to examine the possibility of having other types of boundary conditions allowing for the possibility of prescribing boundary conditions at the singular set. It might be possible to have this type of phenomenon in some classes of diffusive boundary conditions. Is there some example of diffusive boundary condition for which the total energy of the particles is conserved yielding trajectories reaching the singular set in finite time? Is it possible to impose then additional boundary conditions of the types considered in this paper?

We have obtained estimates for the measured valued solutions P using the duality formula (28.1). It should be possible to obtain detailed pointwise estimates for $P(x, v, t)$ outside the

singular set. However, in this paper we have just derived pointwise estimates for the functions φ solving the adjoint problems.

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